PART 1 – INTRODUCTION TO GRAVITATION AND KEPLER'S LAWS

1.1 Learning Objectives

After studying this part, you should be able to:

- Understand the meaning and importance of gravitation.
- Distinguish between gravitation and gravity.
- Understand the historical development of the concept of planetary motion.
- State and explain Kepler's three laws of planetary motion.
- Understand the concept of an orbit and orbital motion.
- Explain the significance of Kepler's laws.
- Apply Kepler's laws to simple numerical and conceptual problems.
- Understand the connection between Kepler's laws and the universal law of gravitation.

1.2 Introduction to Gravitation

Gravitation is the universal force of attraction that exists between any two objects having mass.
Every object in the universe attracts every other object because of its mass.
Examples include:

- The Earth attracts objects towards its surface.
- The Moon revolves around the Earth.
- The Earth and other planets revolve around the Sun.
- Artificial satellites revolve around the Earth.
- The Sun attracts planets and keeps them in their orbits.
- Planets exert gravitational forces on one another.

Thus, gravitation is a **universal interaction**.

1.3 Gravitation and Gravity

The terms **gravitation** and **gravity** are related but are not identical.

Gravitation

Gravitation refers to the **universal force of attraction between any two masses**.

Gravity

Gravity generally refers to the gravitational attraction exerted by a celestial body, such as the Earth, on objects near or on its surface.
Therefore:
Gravitation → universal attraction between masses
Gravity → gravitational attraction of a particular celestial body

1.4 Why Do Planets Revolve Around the Sun?

The planets revolve around the Sun because of the gravitational attraction between the Sun and the planets.
The gravitational force of the Sun provides the required centripetal effect for orbital motion.
Thus, the orbital motion of a planet is the result of the combination of:

- its tendency to move due to its velocity, and
- the gravitational attraction of the Sun.

A planet does not simply fall directly into the Sun because it possesses a tangential velocity.

1.5 Historical Background

The motion of planets was studied systematically by several scientists.
Important contributions include:

- **Nicolaus Copernicus** proposed a heliocentric model in which planets revolve around the Sun.
- **Tycho Brahe** made detailed astronomical observations.
- **Johannes Kepler** used these observations to formulate three laws of planetary motion.
- **Isaac Newton** later explained Kepler's laws using the universal law of gravitation and the laws of motion.

This development established a mathematical description of planetary motion.

1.6 Heliocentric Model

In the heliocentric model:
Sun → approximately at the centre of the planetary system
The planets revolve around the Sun.
The orbits are not generally perfect circles; they are approximately elliptical.
This was a major step towards understanding the actual motion of planets.

1.7 Kepler's Laws of Planetary Motion

Johannes Kepler formulated three laws describing the motion of planets around the Sun.
They are:

1. **Law of Orbits**
2. **Law of Areas**
3. **Law of Periods**

These laws are known collectively as **Kepler's laws of planetary motion**.

1.8 Kepler's First Law – Law of Orbits

Statement

Every planet moves around the Sun in an elliptical orbit, with the Sun at one of the two foci of the ellipse.
Thus, the planetary orbit is generally an **ellipse**, not a perfect circle.
An ellipse has two special points called **foci**.
The Sun occupies one of these foci.

1.9 Important Terms Related to an Elliptical Orbit

Consider a planet moving around the Sun in an elliptical orbit.
Major axis
The longest diameter of the ellipse is called the **major axis**.
Minor axis
The shortest diameter of the ellipse is called the **minor axis**.
Foci
An ellipse has two foci.
The Sun is located at one focus.

Eccentricity
The eccentricity of an ellipse is represented by **e**.
For an ellipse: **0 < e < 1**
For a perfect circle: **e = 0**
Thus, a circular orbit is a special case of an elliptical orbit.

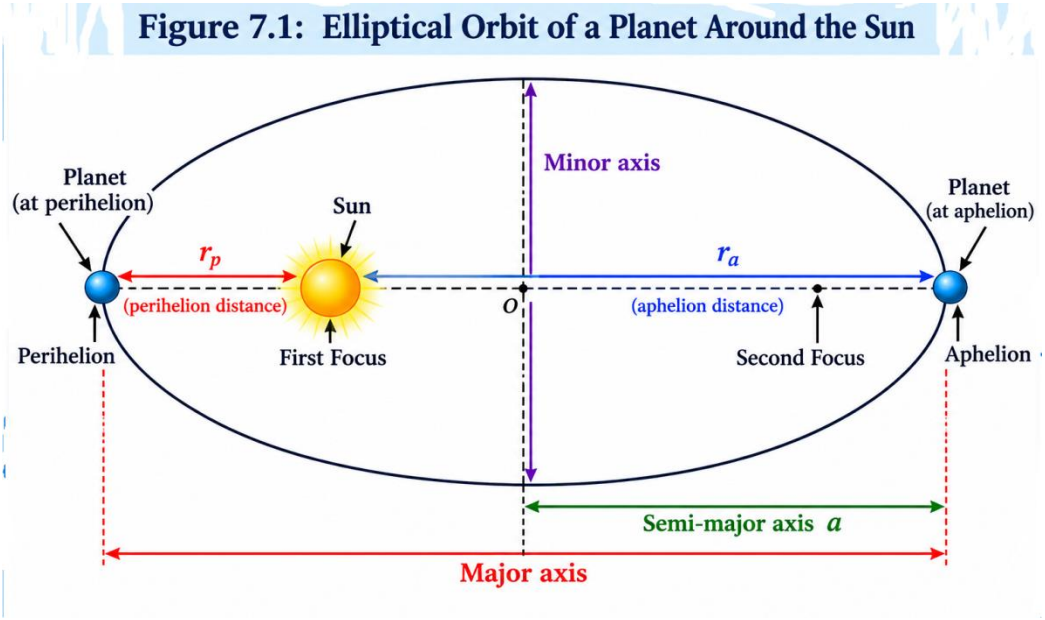
1.10 Perihelion and Aphelion

Since the planetary orbit is elliptical, the distance between the planet and the Sun changes during the orbit.

Perihelion
The point at which the planet is **closest to the Sun** is called **perihelion**.
Distance: $\mathbf{r_p}$

Aphelion
The point at which the planet is **farthest from the Sun** is called **aphelion**.
Distance: $\mathbf{r_a}$
Therefore: $\mathbf{r_p < r_a}$
For an elliptical orbit: $\mathbf{a = (r_p + r_a) /2}$
Where, **a** is the semi-major axis.

Figure 7.1: Elliptical Orbit of a Planet Around the Sun



1.11 Kepler's Second Law – Law of Areas

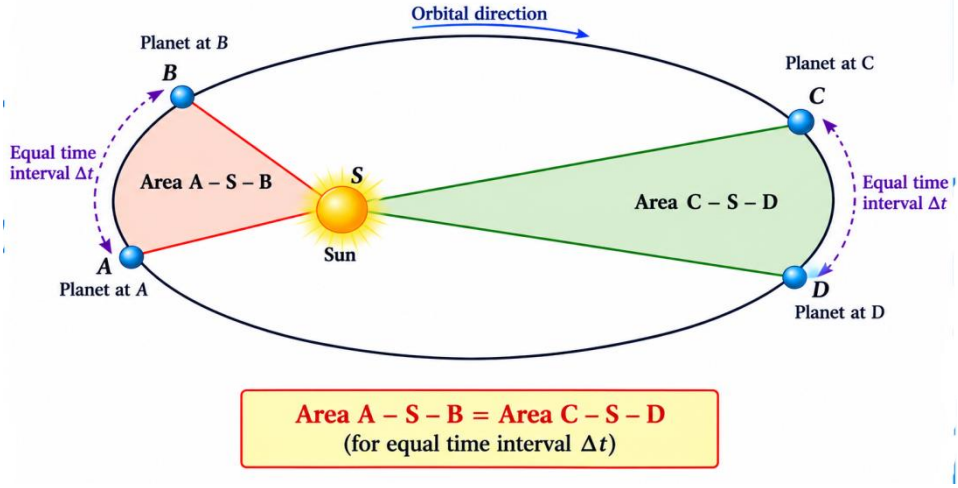
Statement

The line joining the Sun and a planet sweeps out equal areas in equal intervals of time.
This is also called the **law of equal areas**.
Suppose a planet moves from position A to B during a certain time interval.
If it moves from C to D during an equal time interval, then:
Area A-S-B = Area C-S-D
Where, S represents the Sun.

1.12 Meaning of the Law of Areas

The planet does not move with constant speed in an elliptical orbit.
When the planet is closer to the Sun: **Planetary speed is greater.**
When the planet is farther from the Sun: **Planetary speed is smaller.**
Therefore: **Speed at perihelion > Speed at aphelion**
The increased speed near the Sun compensates for the smaller distance, allowing equal areas to be swept in equal times.

Figure 7.2: Equal Area Swept by a Planet in Equal Interval of Time (Kepler's Second Law)



1.13 Consequence of Kepler's Second Law

The second law implies that the areal velocity is constant.
Areal velocity is: **dA/dt = constant**
Where, A is the area swept by the radius vector.
This is closely related to the conservation of angular momentum.
For a planet: **L = constant**
when the gravitational force is central.

1.14 Planetary Speed at Perihelion and Aphelion

At perihelion:

- Distance from Sun is minimum
- Speed is maximum

At aphelion:

- Distance from Sun is maximum
- Speed is minimum

Therefore: $\mathbf{v_p > v_a}$
Where: v_p = speed at perihelion and v_a = speed at aphelion

1.15 Relation at Perihelion and Aphelion

Since angular momentum is conserved: $\mathbf{m\, v_p\, r_p = m\, v_a\, r_a}$
Therefore: $\mathbf{v_p\, r_p = v_a\, r_a}$
Hence: $\mathbf{v_p / v_a = r_a / r_p}$

Since: $\mathbf{r_a > r_p}$
We get: $\mathbf{v_p > v_a}$
This is an important conceptual and numerical relation.

1.16 Kepler's Third Law – Law of Periods

Statement

The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its elliptical orbit.

Mathematically: $\mathbf{T^2 \propto a^3}$
or: $\mathbf{T^2/a^3 = constant}$
Where:

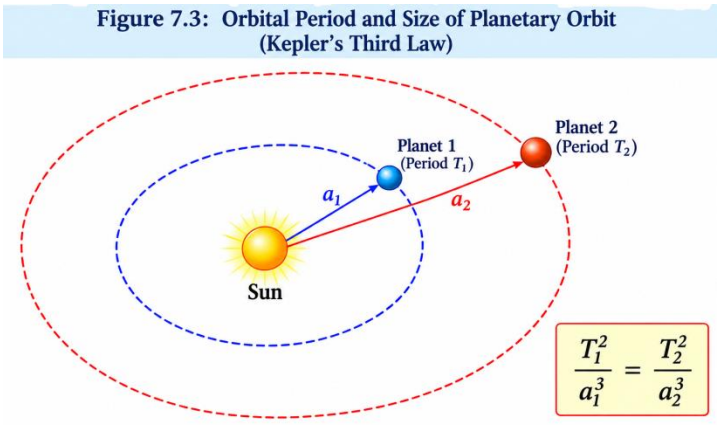
- T = orbital period
- a = semi-major axis of the orbit

For planets revolving around the same Sun: $\mathbf{T_1^2/a_1^3 = T_2^2/a_2^3}$

1.17 For a Circular Orbit

For a circular orbit: $\mathbf{a = r}$
Therefore, Kepler's third law becomes: $\mathbf{T^2 \propto r^3}$
or: $\mathbf{T^2/r^3 = constant}$
Thus: $T^2 \propto r^{3/2}$

Figure 7.3: Orbital Period and Size of Planetary Orbit (Kepler's Third Law)



1.18 Example – Kepler's Third Law

Suppose two planets have orbital radii: $\mathbf{r_1 = r}$ **and** $r_2 = \mathbf{4r}$**.** Find the ratio of their orbital periods.
Using: $\mathbf{T^2 \propto r^3}$
Therefore: $\mathbf{T_2^2/T_1^2 = (r_2/r_1)^3} = 4^3 = 64$
Hence: $\mathbf{T_2/T_1 = 8}$
Therefore: $\mathbf{T_2 = 8T_1}$
Answer: The second planet takes **8 times** as long to complete one revolution.

1.19 Kepler's Laws – Summary

Law	Statement	Mathematical form
First Law	Planet moves in an elliptical orbit with Sun at one focus	Elliptical orbit
Second Law	Equal areas are swept in equal times	dA/dt = constant
Third Law	Square of period is proportional to cube of semi-major axis	T^2/a^3 = constant

1.20 Significance of Kepler's Laws

Kepler's laws provided a mathematical description of planetary motion.
They established that:

- Planetary orbits are elliptical.
- Planetary speed varies during the orbit.
- Planets move faster when closer to the Sun.
- Distant planets take longer to complete an orbit.
- The orbital period is related systematically to the size of the orbit.

Newton later explained these laws using the laws of motion and universal gravitation.

1.21 From Kepler to Newton

Kepler described **how planets move**.
Newton explained **why planets move in this manner**.
The gravitational attraction between the Sun and a planet provides the force responsible for the orbital motion.
This leads to the **universal law of gravitation**, which will be developed in the next part.

1.22 Important Conceptual Connections

Connection 1: Gravitational force → orbital motion
Connection 2: Central force → conservation of angular momentum
Connection 3: Conservation of angular momentum → Kepler's second law
Connection 4: Universal gravitation + Newton's laws → Kepler's laws

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1.23 NEET/JEE Quick Insights

Shortcut 1: For the same central body:

$$T \propto r^{3/2}$$

Shortcut 2: If orbital radius becomes **n times**:

$$T' = n^{3/2}T$$

Shortcut 3: At perihelion: **Speed is maximum**
Shortcut 4: At aphelion: **Speed is minimum**
Shortcut 5: For perihelion and aphelion:

$$v_p r_p = v_a r_a$$

Shortcut 6: Equal areas in equal times imply: **dA/dt = constant**

1.24 Quick Revision Table

Term	Meaning
Gravitation	Universal attraction between masses
Gravity	Gravitational attraction of a celestial body
Ellipse	General shape of planetary orbit
Focus	One of two special points of an ellipse
Perihelion	Nearest point to the Sun
Aphelion	Farthest point from the Sun
Kepler's 1st law	Elliptical orbit
Kepler's 2nd law	Equal areas in equal times
Kepler's 3rd law	$T^2 \propto a^3$
Maximum planetary speed	At perihelion
Minimum planetary speed	At aphelion

PART 2 – UNIVERSAL LAW OF GRAVITATION AND GRAVITATIONAL FORCE

2.1 Learning Objectives

After studying this part, you should be able to:

- State Newton's universal law of gravitation.
- Calculate gravitational force between two masses.
- Understand the meaning and SI unit of gravitational constant G.
- Write the gravitational force in scalar and vector form.
- Apply the principle of superposition to gravitational forces.
- Understand the difference between mass and gravitational force.
- Solve Board, NEET and JEE-level numerical problems involving gravitational force.
- Understand why gravitation is a universal force.

2.2 Newton's Universal Law of Gravitation

Newton proposed that every object in the universe attracts every other object with a force that depends on their masses and the distance between them.

Statement

Every particle in the universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centres.

Consider two masses: $\mathbf{m_1}$ **and** $\mathbf{m_2}$ separated by distance: **r**

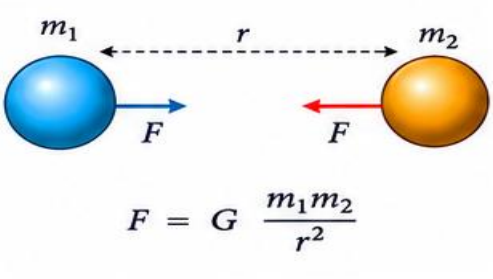
Then the magnitude of gravitational force is: $\mathbf{F \propto m_1 m_2}$ and $\mathbf{F \propto 1/r^2}$

Combining these relations: $\mathbf{F \propto m_1 m_2 / r^2}$

Therefore: $\mathbf{F = G m_1 m_2 / r^2}$

Where, **G** is the universal gravitational constant.

Figure 7.4: Newton's Law of Universal Gravitation



2.3 Nature of Gravitational Force

Gravitational force has several important properties.

1. It is always attractive

Two masses always attract each other.

Unlike electric force, gravitational force does not have repulsive interaction between ordinary positive masses.

2. It is a central force
The force acts along the line joining the centres of the two masses.

3. It is an inverse-square force: $F \propto 1/r^2$

4. It is a long-range force
Gravitational interaction theoretically extends to arbitrarily large distances, although its magnitude decreases rapidly with distance.

5. It is universal: It acts between all objects having mass.

2.4 Gravitational Constant G
The constant **G** is called the **universal gravitational constant**.
Its SI value is approximately: $\mathbf{G = 6.674 \times 10^{-11}\ N\ m^2\ kg^{-2}}$
For many school-level numerical problems: $\mathbf{G \approx 6.67 \times 10^{-11}\ N\ m^2\ kg^{-2}}$
SI Unit: $\mathbf{N\ m^2\ kg^{-2}}$
It can also be written as: $\mathbf{m^3\ kg^{-1}\ s^{-2}}$

2.5 Physical Meaning of G
Consider two masses of: $\mathbf{m_1 = 1\ kg}$ and $\mathbf{m_2 = 1\ kg}$ separated by: $\mathbf{r = 1\ m}$
Then: $\mathbf{F = G(1)(1)/(1)^2}$
Therefore: $\mathbf{F = G}$
Hence, **G is numerically equal to the gravitational force between two point masses of 1 kg each separated by 1 m in vacuum.**
Thus: $\mathbf{G \approx 6.67 \times 10^{-11}\ N}$
The extremely small value of G explains why gravitational attraction between ordinary laboratory objects is very weak.

2.6 Dimensional Formula of G
From: $\mathbf{F = Gm_1m_2/r^2}$
Therefore: $\mathbf{G = Fr^2/(m_1m_2)}$
Dimensions: $[F] = ML\ T^{-2}$; $[r^2] = L^2$; $[m_1m_2] = M^2$
Therefore: $[G] = (ML\ T^{-2})\ (L^2)\ /M^2$
Hence: $\mathbf{[G] = [M^{-1}L^3T^{-2}]}$
Dimensional Formula: $\mathbf{[G] = [M^{-1}\ L^3\ T^{-2}]}$

2.7 Gravitational Force in Vector Form
The gravitational force is a vector quantity. Consider two point masses m_1 and m_2.
The vector form can be written as: $\mathbf{F_{12} = -Gm_1m_2/r^2\ \hat{r}_{12}}$
The negative sign indicates that the force is attractive.
If $\hat{r}_{12}$ is a unit vector directed from m_1 towards m_2, then the force on m_1 due to m_2 acts in the opposite direction.
Similarly: $\mathbf{F_{21} = -F_{12}}$
This is consistent with Newton's third law.

2.8 Gravitational Force and Newton's Third Law
The gravitational forces between two masses are equal in magnitude and opposite in direction.

Therefore: $\mathbf{F_{12} = -F_{21}}$
Hence: $\mathbf{|F_{12}| = |F_{21}|}$
These forces act on different bodies.
Therefore, they form an **action-reaction pair**.

2.9 Gravitational Force Depends on Mass

From: $\mathbf{F = Gm_1m_2/r^2}$
We obtain: $\mathbf{F \propto m_1}$ and: $\mathbf{F \propto m_2}$
Therefore, if one mass is doubled while everything else remains unchanged: **F' = 2F**
If both masses are doubled: **F' = 4F**

2.10 Dependence on Distance

From: $\mathbf{F \propto 1/r^2}$
Therefore: **If distance is doubled: F' = F/4**
If distance is tripled: F' = F/9
If distance becomes half: F' = 4F
This inverse-square dependence is very important in numerical problems.

2.11 Example 1 – Effect of Changing Masses

Two masses experience gravitational force F.
If both masses are doubled while their separation remains unchanged, find the new force.
Original: $\mathbf{F = [Gm_1m_2] /r^2}$
New masses: $\mathbf{m_1' = 2m_1}$ and $\mathbf{m_2' = 2m_2}$
Therefore: $\mathbf{F' = [G (2m_1) (2m_2)] /r^2}$ **Or** **F' = 4F**
Answer: New force = 4F

2.12 Example 2 – Effect of Changing Distance

If the distance between two masses is increased from r to 3r, what happens to the gravitational force?
Since: $\mathbf{F \propto 1/r^2}$
Therefore: $\mathbf{F' = F/3^2}$
Hence: **F' = F/9**
Answer: The force becomes **one-ninth** of its original value.

2.13 Superposition Principle for Gravitation

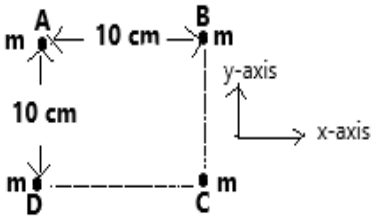
If several masses are present, the net gravitational force on a particular mass is the **vector sum of the gravitational forces due to all the other masses individually**.
If a mass experiences forces: $\mathbf{F_1, F_2, F_3}$ **...** then: $\mathbf{F_{net} = F_1 + F_2 + F_3 +...}$
Or: $\mathbf{F_{net} = \Sigma F_i}$
Since gravitational force is a vector, vector addition must be used.

2.14 Gravitational Force Due to Three Masses

Suppose a mass m is acted upon by three masses: $\mathbf{m_1, m_2}$ **and** $\mathbf{m_3}$
Then: $\mathbf{F_{net} = F_1 + F_2 + F_3}$

Each force is calculated independently using: $\boldsymbol{F = Gm_i m/}$
The vectors are then added.
For forces along one straight line, ordinary algebraic addition with signs can be used.
For forces in different directions, vector addition is required.

For example,



To find net force at point A on mass m as:

Procedures:

First: Show the direction of each force on mass m at point A.
Second: Write the magnitude of each force acting at point A.
Third: Resolve each force in horizontal and vertical direction as:

$$F_x \text{ and } F_y$$

Fourth: Find net magnitude of force as:

$$F_A = F_{net} = \sqrt{{F_x}^2 + {F_y}^2}$$

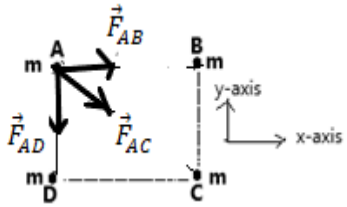
Fifth: Find the direction of net force as

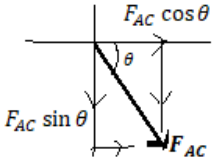
$$\tan\varphi = \frac{F_y}{F_x}$$

Or,

$$\varphi = tan^{-1}\left(\frac{F_y}{F_x}\right)$$

Methods of solution of this question:





Solution: First Method

Three forces acting at point A: Force between two masses is attractive. Therefore, force at a point will have direction towards other mass.

Direction of three forces at point A: It is shown in the adjoining figure.

$\vec{F}_{AB}$: Along the line AB, towards point B mass
$\vec{F}_{AD}$: Along the line AD, towards point D mass
$\vec{F}_{AC}$: Along the AC, towards point C mass

Magnitude of each force:

Using:

$$F = G\frac{m_1 m_2}{r^2}$$

Let AB = BC = CD = DA = r and AC = a = $r\sqrt{2}$

$$F_{AB} = \frac{Gm^2}{r^2} = F\ (say)$$

$$F_{AD} = \frac{Gm^2}{r^2} = F$$

$$F_{AC} = \frac{Gm^2}{a^2} = \frac{Gm^2}{2r^2} = \frac{F}{2}$$

Resolving $\vec{F}_{AC}$ in x-direction and y-direction as shown: we get (using sign +ve or –ve)

$$F_x = F_{AB} + F_{AC}\cos\theta = \frac{Gm^2}{r^2} + \frac{Gm^2}{2r^2}\cos 45^0 = \frac{Gm^2}{r^2}\left(1 + \frac{1}{2\sqrt{2}}\right)$$

Similarly,

$$F_y = -F_{AD} - F_{AC}\sin\theta = -\frac{Gm^2}{r^2} - \frac{Gm^2}{2r^2}\sin 45^0 = -\frac{Gm^2}{r^2}\left(1 + \frac{1}{2\sqrt{2}}\right)$$

Therefore, net force at point A:

$$F_A{}^2 = F_x{}^2 + F_y{}^2 = \left(\frac{Gm^2}{r^2}\right)^2\left[1 + \frac{1}{8} + \frac{1}{\sqrt{2}} + 1 + \frac{1}{8} + \frac{1}{\sqrt{2}}\right] = (2 + \frac{1}{4} + \sqrt{2})\left(\frac{Gm^2}{r^2}\right)^2$$

Or

Hence, net force at point A above in the figure is:

$$F_A = \frac{Gm^2}{r^2}\sqrt{\left(\sqrt{2} + \frac{1}{2}\right)^2} = \frac{Gm^2}{r^2}\left(\sqrt{2} + \frac{1}{2}\right)$$

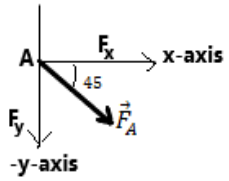
Therefore, the magnitude of net force at point A:

$$\boxed{F_A = \frac{Gm^2\left(2\sqrt{2} + 1\right)}{2r^2}}$$

Putting the value of m = 10 kg; r = 10 cm = 0.1 m; G =$6.67 \times 10^{-11} Nm^2kg^{-2}$; we get

$F_A = \frac{6.67\times10^{-11}\times100}{10^{-2}}\left(\frac{2\sqrt{2}+1}{2}\right) = 3.335 \times 10^{-7}\left(2\sqrt{2} + 1\right)N \approx 6.16 \times 10^{-7}N$

We obtain,



Answer:

$$\boxed{\mathrm{F_A} \approx 6.16 \times 10^{-7}\ \mathrm{N}}$$

Direction of net force at point A:

$$\tan\varphi = \frac{F_y}{F_x} = -1 = \tan 45^0 \quad Or\ \varphi = -45^0$$

Answer: $6.16 \times 10^{-7}N$

Second Method:

Using Unit vector method (taking unit vector $\hat{\imath}$ along x −axis and $\hat{\jmath}$ along y-axis)

Solution:

Net force at point A

Magnitude of each force:

$$F_{AB} = \frac{Gm^2}{r^2} = F\ (say)$$

$$F_{AD} = \frac{Gm^2}{r^2} = F$$

$$F_{AC} = \frac{Gm^2}{a^2} = \frac{Gm^2}{2r^2} = \frac{F}{2}$$

Therefore,

$$\vec{F}_A = \vec{F}_{AB} + \vec{F}_{AD} + \vec{F}_{AD} = F\hat{\imath} + F(-\hat{\jmath}) + (F/2)\sin 45^0\,(-\hat{\jmath}) + (F/2)\cos 45^0\,(\hat{\imath})$$

Or,

$$\vec{F}_A = F\left[\left(1 + \frac{1}{2\sqrt{2}}\right)\hat{\imath} - \left(1 + \frac{1}{2\sqrt{2}}\right)\hat{\jmath}\right]$$

Or,

$$\vec{F}_A = \frac{F}{2\sqrt{2}}\left(2\sqrt{2} + 1\right)(\hat{\imath} - \hat{\jmath})$$

Therefore, we get the net force at point A in vector form:

$$\boxed{\vec{F}_A = \frac{Gm^2}{2\sqrt{2}r^2}\left(2\sqrt{2} + 1\right)(\hat{\imath} - \hat{\jmath})}$$

In magnitude form, the net force at point A:

$$\boxed{F_A = \frac{Gm^2\left(2\sqrt{2} + 1\right)}{2r^2}}$$

Putting the value of m = 10 kg; r = 10 cm = 0.1 m; G =$6.67 \times 10^{-11} Nm^2kg^{-2}$; we get

$$F_A = \frac{6.67 \times 10^{-7}}{2\sqrt{2}}\left(2\sqrt{2} + 1\right)(\hat{\imath} - \hat{\jmath})N = \frac{3.35 \times 10^{-7}}{\sqrt{2}}\left(2\sqrt{2} + 1\right)(\hat{\imath} - \hat{\jmath})N$$

Answer: $\frac{3.35\times10^{-7}}{\sqrt{2}}\left(2\sqrt{2} + 1\right)(\hat{\imath} - \hat{\jmath})N$

Third Method:
Using law of parallelogram method:
Solution:
Since,

$$F_{AB} = \frac{Gm^2}{r^2} = F\ (say)$$

$$F_{AD} = \frac{Gm^2}{r^2} = F$$

$$F_{AC} = \frac{Gm^2}{a^2} = \frac{Gm^2}{2r^2} = \frac{F}{2}$$

The angle between $\vec{F}_{AB}$ and $\vec{F}_{AD}$is $\theta = 90^0$, and let F' be its resultant force, then,

$$F' = \sqrt{F^2 + F^2 + 2F^2\cos 90^0} = F\sqrt{2}$$

Since, $F_{AB} = F_{AD}$, then $\vec{F}'$ will also be along $\vec{F}_{AC}$; therefore,

$$F_A = F_{AC} + F' = \frac{F}{2} + F\sqrt{2}$$

Or,

$$F_A = \frac{F}{2}\left(2\sqrt{2} + 1\right)$$

Therefore, the magnitude of net force at point A:

$$\boxed{F_A = \frac{Gm^2\left(2\sqrt{2} + 1\right)}{2r^2}}$$

Putting the value of m = 10 kg; r = 10 cm = 0.1 m; G =$6.67 \times 10^{-11} Nm^2 kg^{-2}$; we get

$$F_A = \frac{Gm^2}{2r^2}\left(2\sqrt{2}+1\right) = 3.335 \times 10^{-7}\left(2\sqrt{2}+1\right)N$$

Answer: $3.335 \times 10^{-7}\left(2\sqrt{2}+1\right)N$

2.15 Gravitational Force between Extended Bodies

Newton's law in the simple form: $\mathbf{F = Gm_1m_2/r^2}$ directly applies to **point masses**.
For extended bodies, the distance r must be interpreted carefully.
For **spherically symmetric bodies**, the gravitational field outside the body is equivalent to that produced by a point mass concentrated at its centre.
This result is particularly important for planets and stars.

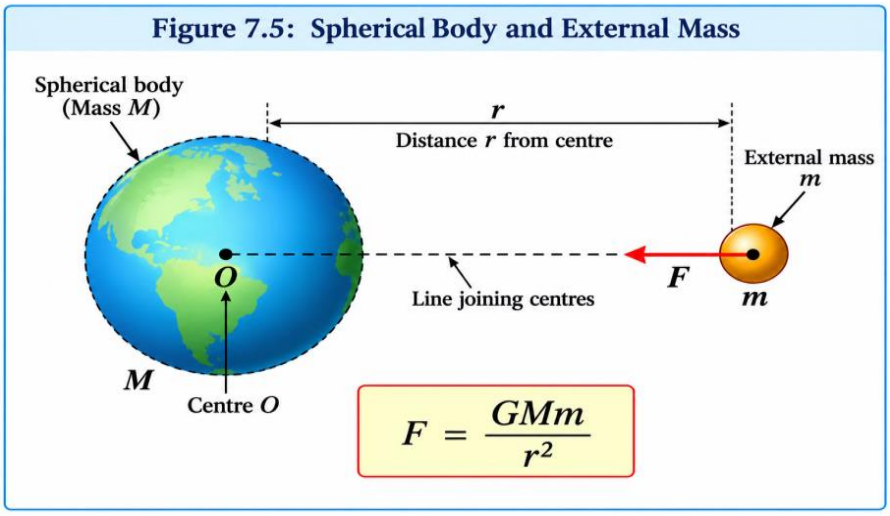
2.16 Spherical Bodies

For a spherically symmetric body of mass M, at an external point located at distance r from its centre: $\mathbf{F = GMm/r^2}$
Where:

- M = mass of the spherical body
- m = external object
- r = distance from the centre of the spherical body

This allows planets to be treated approximately as point masses when studying external gravitational effects.

Figure 7.5: Spherical Body and External Mass



2.17 Gravitational Force Near the Earth

Gravitational force acting on the body of mass m near the earth's surface is its weight.

$$\mathbf{W = mg}$$

According to Newton's law of gravitation,

$$F = \frac{GMm}{R^2}$$

Where: M = mass of Earth and R = radius of Earth
Since, gravitational force equals the weight near the earth's surface:

$$mg = \frac{GMm}{R^2}$$

Canceling m both sides, we get:

$$\boxed{g = \frac{GM}{R^2}}$$

This relation will be developed in detail in the next part.

2.18 Important Difference Between G and g

Students frequently confuse **G** and **g**.

Quantity	G	g
Meaning	Universal gravitational constant	Acceleration due to gravity
Value	6.674×10^{-11} N m^2 kg^{-2}	Approximately 9.8 m s^{-2} near Earth's surface
Nature	Universal constant	Depends on location
Dimension	$M^{-1}L^3T^{-2}$	LT^{-2}
Depends on	No location dependence	Mass and radius of celestial body, and location

Important: G is universal. g is not universal.

2.19 Why is Gravitational Force So Weak?

The value of G is very small: **$G \approx 6.674 \times 10^{-11}$ N m^2 kg^{-2}**
Therefore, gravitational attraction between ordinary objects is usually very small.
However, astronomical bodies such as planets and stars have enormous masses.
Consequently, gravitational forces become significant on astronomical scales.

2.20 Gravitational Force as a Central Force

A force is called a **central force** if:

1. It acts along the line joining the two interacting bodies.
2. Its magnitude depends only on their separation.

Gravitational force satisfies both conditions.
Therefore, gravity is a **central force**.
This property is important in understanding:

- Planetary motion
- Satellite motion
- Conservation of angular momentum
- Orbital motion

2.21 Gravitational Force and Orbital Motion

Consider a planet of mass m moving around a star of mass M.
The gravitational force is: **$F = GMm/r^2$**
This force acts towards the star.
For a circular orbit, the required centripetal force is: **$F_C = mv^2/r$**
Therefore: **$GMm/r^2 = mv^2/r$**
Cancelling m: **$GM/r = v^2$**
Hence:

$$\boxed{v_0 = \sqrt{\frac{GM}{r}}}$$

This is the **orbital velocity for a circular orbit**.
A detailed treatment of orbital velocity will be given later in this chapter.

2.22 Example 3 – Gravitational Force

Two bodies of masses: $\mathbf{m_1 = 10\ kg}$ and $\mathbf{m_2 = 20\ kg}$ are separated by: $\mathbf{r = 2\ m}$
Find the gravitational force.
Using: $\mathbf{F = Gm_1m_2/r^2}$ $F = (6.67 \times 10^{-11})(10)(20)/(2)^2 = (6.67 \times 10^{-11})(200)/4$
Therefore: $\mathbf{F = 3.335 \times 10^{-9}\ N}$
Answer: $\mathbf{F \approx 3.34 \times 10^{-9}\ N}$

2.23 Example 4 – Ratio of Gravitational Forces

Two pairs of masses have the same separation. For the first pair: $\mathbf{m_1 = m, m_2 = m}$
For the second pair: $\mathbf{m_1 = 2m, m_2 = 3m}$
Find the ratio of the gravitational forces.
First: $\mathbf{F_1 = Gm^2/r^2}$
Second: $\mathbf{F_2 = G(2m)(3m)/r^2 = 6Gm^2/r^2}$
Therefore: $\mathbf{F_2/F_1 = 6}$
Answer: $\mathbf{F_2 : F_1 = 6 : 1}$

2.24 Quick Revision Table

Concept	Key Point
Universal law	$F = Gm_1m_2/r^2$
Nature	Attractive
Force type	Central force
G	Universal gravitational constant
Value of G	$6.674 \times 10^{-11}\ N\ m^2\ kg^{-2}$
Dimension of G	$M^{-1}L^3T^{-2}$
Superposition	Vector sum of individual forces
External spherical body	Acts as if mass concentrated at centre
Near Earth	$g = GM/R^2$
Circular orbital velocity	$v_0 = \sqrt{GM/r}$

PART 3 – ACCELERATION DUE TO GRAVITY AND VARIATION OF g

3.1 Learning Objectives

After studying this part, you should be able to:

- Define acceleration due to gravity.
- Derive the relation between **G and g**.
- Understand the dependence of g on the mass and radius of a celestial body.
- Calculate the value of g near the surface of the Earth.
- Understand the variation of g with altitude.
- Understand the variation of g with depth.
- Explain why g is maximum near the Earth's surface.
- Solve Board, NEET and JEE-level numerical problems involving g.

3.2 Acceleration Due to Gravity

When an object is released near the surface of the Earth, it falls towards the Earth because of Earth's gravitational attraction.

The acceleration produced in the object due to this gravitational force is called **acceleration due to gravity**.

It is represented by: **g**

Near the Earth's surface: $\mathbf{g \approx 9.8\ m\ s^{-2}}$

The direction of g is towards the centre of the Earth.

3.3 Gravitational Force on a Body Near Earth

Consider a body of mass **m** near the surface of the Earth.

Let:

- M = mass of Earth
- R = radius of Earth
- m = mass of the body

According to Newton's law of gravitation: $\mathbf{F = GMm/R^2}$

The weight of the body is: **W = mg**

Since gravitational force produces the weight: $\mathbf{mg = GMm/R^2}$

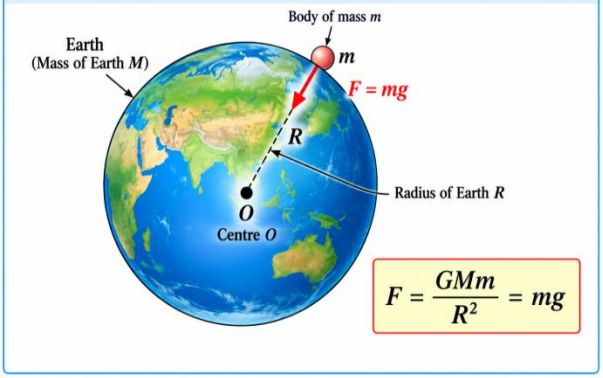
Cancel m: $\mathbf{g = GM/R^2}$

Therefore,

$$\mathbf{g = GM/R^2}$$

This is one of the most important equations in Gravitation.

Figure 7.6: Gravitational Force on a Body at the Earth's Surface



3.4 Relation between G and g

From: $\mathbf{g = GM/R^2}$

We can see that acceleration due to gravity depends on:

- Mass M of the Earth
- Radius R of the Earth
- Universal gravitational constant G

Thus: $\mathbf{g \propto M}$ and $\mathbf{g \propto 1/R^2}$ for different spherical celestial bodies.

3.5 Important Consequence

For two planets: $\mathbf{g_1 = GM_1/R_1^2}$ and $\mathbf{g_2 = GM_2/R_2^2}$
Therefore: $\mathbf{g_1/g_2 = (M_1/M_2)\ (R_2^2/R_1^2)}$
This relation is useful for comparing the surface gravity of different planets.

3.6 Mass Does Not Affect g for a Freely Falling Body

From: **F = mg** and: $\mathbf{F = GMm/R^2}$
We obtain: $\mathbf{g = GM/R^2}$
Notice that the mass **m of the falling body cancels out**.
Therefore, neglecting air resistance:
All bodies fall with the same acceleration at a given location.
This is an important consequence of the equivalence of gravitational and inertial mass in Newtonian mechanics.

3.7 Weight of a Body

The weight of an object is the gravitational force acting on it.
Therefore: **W = mg**
Where: W = weight, m = mass, g = acceleration due to gravity
The SI unit of weight is: **newton (N)**
Mass is measured in kilograms.

Important Difference

Mass → kg
Weight → N
Mass remains constant for a given object, whereas weight can change when g changes.

3.8 Factors Affecting g

The value of g depends mainly on:

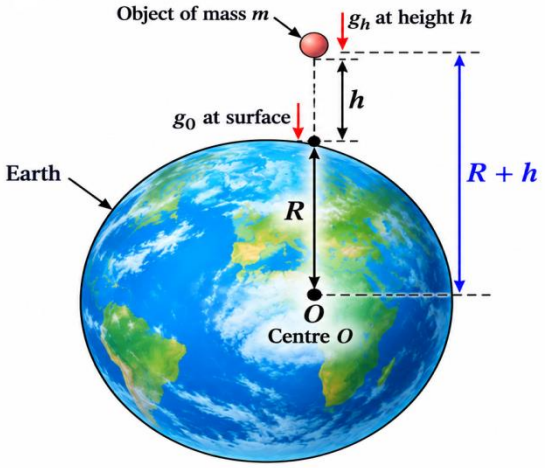
1. Mass of the celestial body
2. Radius of the celestial body
3. Distance from the centre of the celestial body

For Earth, g also varies slightly with latitude because of Earth's rotation and its non-perfectly spherical shape. This effect will be discussed briefly later.

3.9 Variation of g with Altitude

Consider a body at a height **h** above the Earth's surface.
The Earth's radius is **R**.
Therefore, the distance of the body from the Earth's centre is: **r = R + h**
Using the gravitational relation: $\mathbf{g' = GM/(R+h)^2}$
At Earth's surface: $\mathbf{g = GM/R^2}$
Dividing: $\mathbf{g' /g = R^2/(R+h)^2}$
Therefore: $\mathbf{g' = g\ [R/(R+h)]^2}$
This is the exact expression for acceleration due to gravity at height h.

Figure 7.7: Acceleration due to Gravity at Height *h*



$$g_h = \frac{GM}{(R+h)^2} = g_0\left(\frac{R}{R+h}\right)^2$$

$$g_0 = \frac{GM}{R^2}$$

3.10 For Small Heights

If: **h << R** then:

$$g' = \frac{g}{\left(1+\frac{h}{R}\right)^2}$$

can be approximated using: $\mathbf{(1+x)^{-2} \approx 1 - 2x}$ for x much smaller than 1.
Therefore:

$$\boxed{g' = g\left(1-\frac{2h}{R}\right)}$$

Thus:

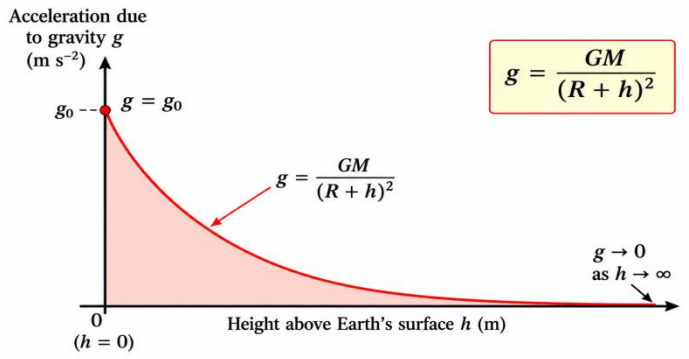
$$\boxed{\frac{\Delta g}{g} = -\frac{2h}{R}}$$

This shows that g decreases approximately linearly with height when h is small compared with R.

3.11 Graph of g with Height

As height increases: **g decreases.**
At: **h = 0,** we have: $\mathbf{g = g_0}$
As h becomes very large: $\mathbf{g \rightarrow 0}$ because: $\mathbf{g' = GM/(R+h)^2}$ and the denominator becomes very large.

Figure 7.8: Variation of *g* with Height Above Earth's Surface



$$g = \frac{GM}{(R+h)^2}$$

3.12 Example – g at a Height Equal to Earth's Radius

Suppose: **h = R** Then: **g' = g[R/(R+R)]² Or, g' = g(R/2R)²**
Therefore: **g' = g/4**
Answer: At a height equal to Earth's radius, acceleration due to gravity is: **g/4**

3.13 Approximate Height Formula

For small height h: **g' ≈ g(1 − 2h/R)**
Therefore: **g – g' ≈ 2gh/R**
This is useful for small-altitude approximation problems.

3.14 Variation of g with Depth

Now consider a body placed at a depth **d** below the Earth's surface.
Let:

- R = Earth's radius
- d = depth
- r = distance from Earth's centre

Then: **r = R − d**
Assuming Earth has uniform density, the mass enclosed within radius r is proportional to r^3.
Therefore: **M' = M (r^3/R^3)**
Where, M is the total mass of Earth.
The gravitational acceleration at radius r is: **g' = GM'/r^2**
Substituting:

$$g' = \left(\frac{G}{r^2}\right)\left(\frac{Mr^3}{R^3}\right)$$

Therefore:

$$g' = \frac{GMr}{R^3}$$

Using: $g = GM/R^2$ and by substituting $r = R - d$, we obtain:

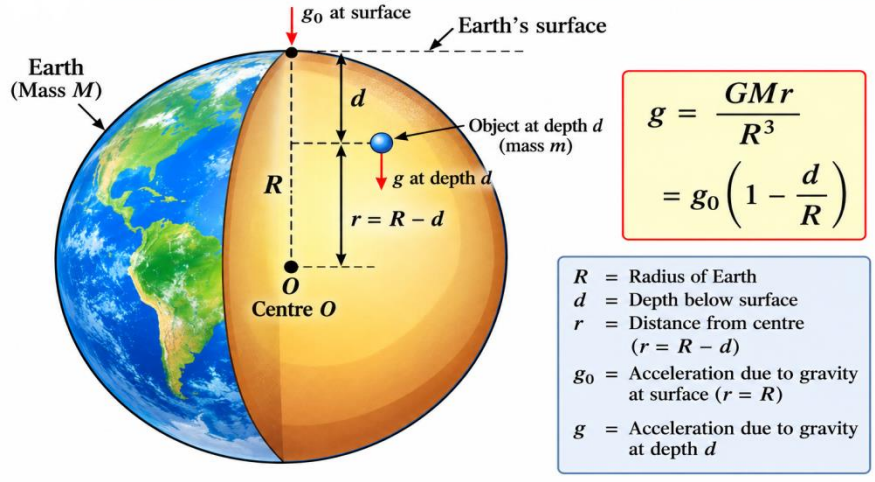
$$g' = \frac{gr}{R} = g\left(\frac{R-d}{R}\right) = g\left(1 - \frac{d}{R}\right)$$

Therefore, we get:

$$\boxed{g' = g\left(1 - \frac{d}{R}\right)}$$

This is the standard expression for g at depth d inside a uniform Earth.

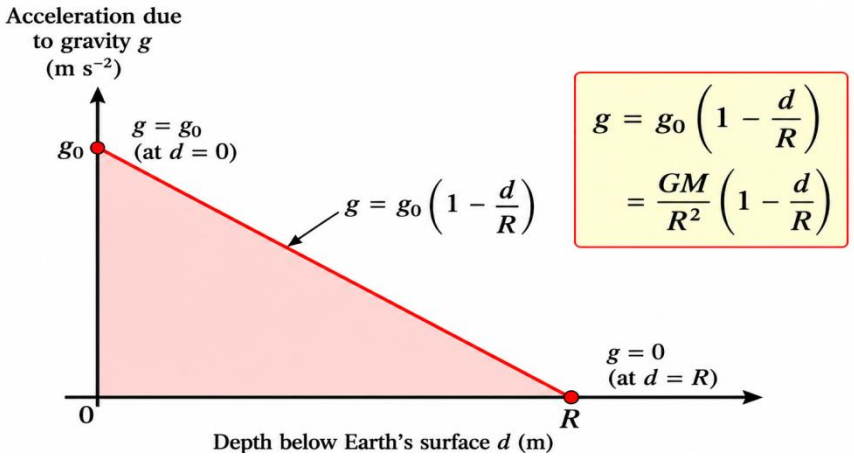
Figure 7.9: Acceleration Due to Gravity at Depth *d*



3.15 Graph of g with Depth

According to: **g' = g(1 − d/R);** g decreases linearly with depth.
At the surface: **d = 0,** Therefore: **g' = g**
At the centre: **d = R**
Therefore: **g' = g(1 − 1)**
Hence: **g = 0**
Thus, for the ideal uniform-Earth model: **g decreases linearly from g at the surface to zero at the centre.**

Figure 7.10: Variation of *g* with Depth below Earth's Surface



3.16 Comparison of Variation with Height and Depth

Condition	Expression for g
At surface	$g = GM/R^2$
Height h	$g' = g[R/(R+h)]^2$
Small height	$g' \approx g(1-2h/R)$
Depth d	$g' = g(1-d/R)$
At Earth's centre	$g = 0$

Important Difference

For small height: **g decreases approximately as 1 − 2h/R**
For depth: **g decreases linearly as 1 − d/R;** under the uniform-density Earth approximation.

3.17 At the Centre of the Earth

At the centre: **r = 0**
By spherical symmetry, gravitational attractions from all directions cancel.
Therefore: **g = 0**
This does **not** mean that gravitational interaction has disappeared everywhere around the centre. Rather, the vector contributions cancel at the exact centre.

3.18 Variation of g with Latitude

The Earth rotates about its axis.
Because of this rotation, the effective value of g varies with latitude.
The effective value of g is:

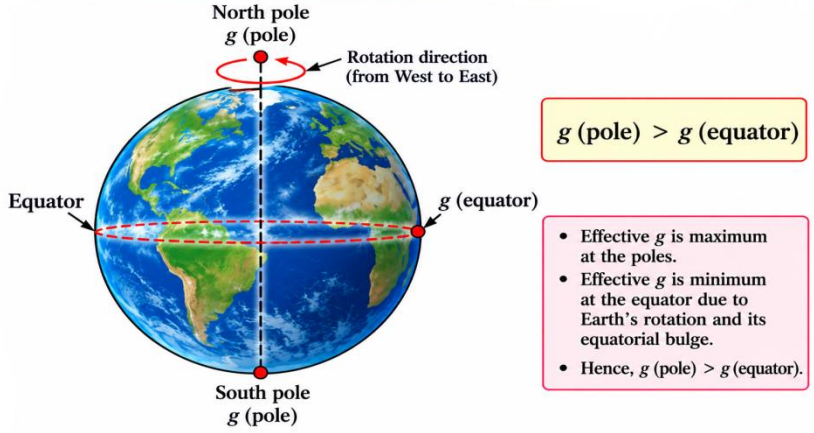
- smaller near the equator,
- larger near the poles.

Therefore: **g (equator) < g (pole)**
This is partly due to:

1. Earth's rotation.
2. Earth's slightly non-spherical shape.

At the poles, the effect of Earth's rotation on effective g is zero because the distance from the rotation axis is zero.
At the equator, the rotational effect is maximum.

Figure 7.11 (a): Variation of Effective g from Equator to Pole:



3.19 Effect of Earth's Rotation

For a body at latitude λ, the effective acceleration due to gravity is influenced by Earth's angular velocity.
The simplified relation is:

$$\boxed{g_{eff} = g - \omega^2 R \cos^2 \lambda}$$

for a spherical earth approximation.

Where:

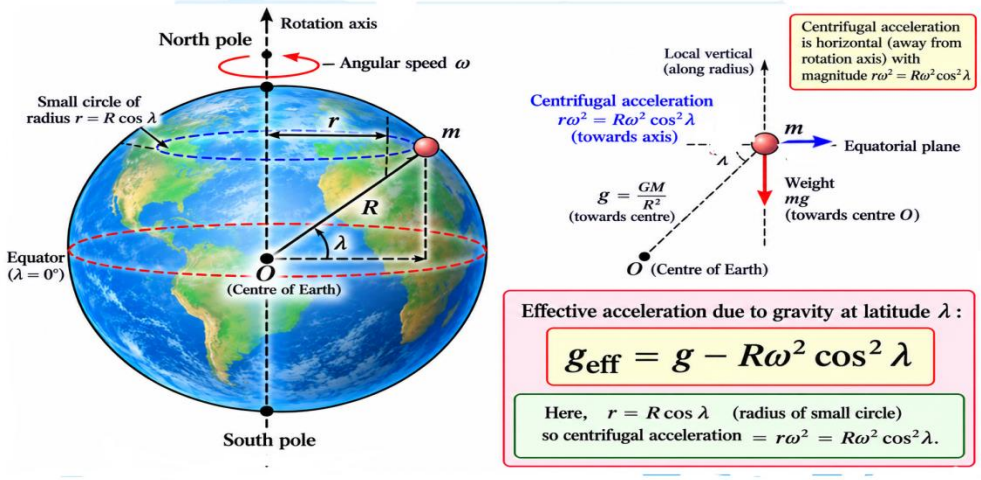
- g= gravitational acceleration at a place before the effect of earth's rotation
- ω = Angular velocity of earth
- R = Radius of earth
- λ = latitude angle

At the equator: **$\lambda = 0°$**
Therefore:

$$g_{eff} = g - \omega^2 R$$

At the poles: **$\lambda = 90°$**
Therefore: **$g_{eff} = g$**
This explains why effective g is smaller at the equator.

Figure 7.11 (b): Variation of g due to Earth's Rotation with Lattitude Angle



3.20 Important Numerical Example – Height

At what height above Earth's surface will g become one-fourth of its surface value?
Given: **g' = g/4**
Using: **g' = g[R/(R+h)]²**
Therefore: **1/4 = [R/(R+h)]²**
Taking square root: **1/2 = R/(R+h)**
Therefore: **R+h = 2R**
Hence: **h = R**
Answer: The required height is equal to the Earth's radius.

3.21 Important Numerical Example – Depth

At what depth inside the Earth will g become half its surface value?
Given: **g' = g/2**
Using: **g' = g (1−d/R)**
Therefore: **1/2 = 1−d/R**
Hence: **d/R = 1/2**
Therefore: **d = R/2**
Answer: The required depth is **R/2**.

3.22 Example – Comparing g at Height and Depth

Suppose: **h = R/2**
At this height: **g' = g[R/(R+R/2)]² = g[R/(3R/2)]²**
Therefore: **g' = 4g/9**
Now consider a depth: **d = R/2**
Then: **g' = g (1−1/2)**
Therefore: **g' = g/2**
Thus: **g' = 4g / 9 and** g' **= g/2**
This illustrates that g does not decrease in exactly the same way with equal numerical height and depth.

3.23 Quick Revision Table

Situation	Acceleration due to gravity
Earth's surface	$g = GM/R^2$
Height h	$g[R/(R+h)]^2$
Small height h	$g(1-2h/R)$
Depth d	$g(1-d/R)$
Earth's centre	0
Pole	Relatively maximum effective g
Equator	Relatively minimum effective g

Master Concept

The acceleration due to gravity at the Earth's surface is $g = GM/R^2$; it decreases with altitude and, under the uniform-density Earth approximation, decreases linearly with depth, becoming zero at the Earth's centre.

PART 4 – GRAVITATIONAL FIELD, POTENTIAL, POTENTIAL ENERGY, BINDING ENERGY AND ESCAPE VELOCITY

4.1 Learning Objectives

After studying this part, you should be able to:

- Define gravitational field and gravitational field intensity.
- Calculate gravitational field due to a point mass.
- Understand gravitational potential.
- Calculate gravitational potential due to a point mass and a system of masses.
- Understand gravitational potential energy.
- Derive the work done in raising a body from Earth's surface to a height h.
- Understand the difference between gravitational potential and gravitational potential energy.
- Explain gravitational binding energy.
- Derive the expression for escape velocity.
- Solve Board, NEET and JEE-level numerical problems based on these concepts.

4.2 Gravitational Field

A mass produces a gravitational influence in the space surrounding it.
The region around a mass in which another mass experiences a gravitational force is called the **gravitational field**.
For example, the Earth produces a gravitational field around itself.
A body placed in this field experiences gravitational attraction towards the Earth.

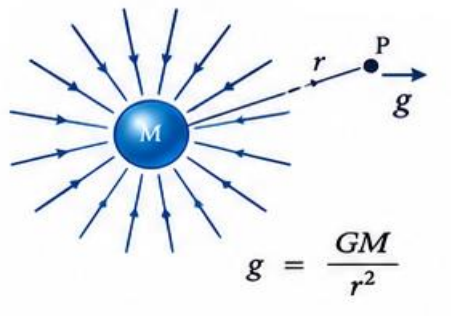
4.3 Gravitational Field Intensity

The **gravitational field intensity** at a point is defined as the gravitational force experienced by a unit mass placed at that point.
If a test mass m experiences gravitational force F, then: $\mathbf{g = F/m}$
The gravitational field intensity is a vector quantity.
Its direction is the direction of the gravitational force on a unit positive test mass.
SI Unit: $\mathbf{N\ kg^{-1}}$
Since: $\mathbf{1\ N\ kg^{-1} = 1\ m\ s^{-2}}$
The SI unit can also be written as: $\mathbf{m\ s^{-2}}$

4.4 Gravitational Field Due to a Point Mass

Consider a point mass M.
A small test mass m is placed at distance r from M.
The gravitational force is: $\mathbf{F = GMm/r^2}$
Therefore, gravitational field intensity is: $\mathbf{g = F/m}$
Hence: $\mathbf{g = GM/r^2}$
The direction of the field is **towards M**,
because gravitational force is attractive.

Figure 7.12: Gravitational Field Due to a Point Mass



4.5 Gravitational Field Due to Earth

For an object at distance r from the centre of Earth: $\mathbf{g = GM/r^2}$
At the Earth's surface: $\mathbf{r = R}$
Therefore: $\mathbf{g = GM/R^2}$
At a height h: $\mathbf{r = R+h}$
Therefore: $\mathbf{g' = GM/(R+h)^2}$

4.6 Principle of Superposition of Gravitational Fields

If several masses are present, the net gravitational field at a point is the vector sum of the fields produced by the individual masses.
Thus: $\mathbf{g_{net} = g_1 + g_2 + g_3 + ...}$
Or: $\mathbf{g_{net} = \Sigma g_i}$
Since gravitational field is a vector quantity, vector addition must be used.

4.7 Gravitational Potential

Gravitational potential at a point is defined as the gravitational potential energy per unit mass at that point, taking the potential to be zero at infinity.
For a point mass M, the gravitational potential at distance r is:

$$V = -\frac{GM}{r}$$

Equivalently, it is the work done per unit mass by an external agent in bringing a test mass slowly from infinity to that point, with no change in kinetic energy . This signed work is negative for an attractive gravitational field.
The gravitational potential is represented by **V**.
Thus: $V = W/m_0$
Where W is the work done in bringing a test mass m_0 from infinity to the point.
The reference level is usually taken as: **V = 0 at infinity**

4.8 Gravitational Potential Due to a Point Mass

Consider a point mass M.
At a distance r from M, the gravitational potential is:

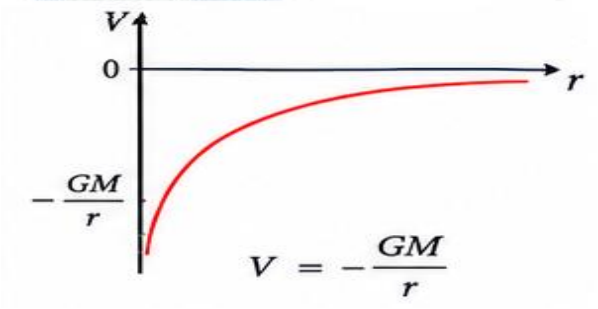
$$V = -\frac{GM}{r}$$

The negative sign indicates that gravitational potential is lower than its value at infinity. A external agent must supply positive work to move a mass slowly from a finite distance to infinity.

4.9 Why is Gravitational Potential Negative?

We choose: **V = 0 at infinity**
A mass at a finite distance r is gravitationally bound to M.
To take it from r to infinity, positive external work must be supplied.
Therefore, the potential at r is negative.
Hence: $\mathbf{V(r) < 0}$ and: $\mathbf{V(\infty) = 0}$
As r increases: $\mathbf{V \rightarrow 0}$ from the negative side.

Figure 7.13: Gravitational Potential due to a Point Mass



4.10 Gravitational Potential Due to Several Masses

Since gravitational potential is a scalar quantity, potentials due to different masses are added algebraically.

For masses: $\mathbf{M_1, M_2, M_3 \ldots}$ at distances: $\mathbf{r_1, r_2, r_3, \ldots}$ from a point:

$$\mathbf{V = -GM_1/r_1 - GM_2/r_2 - GM_3/r_3 - \ldots}$$

Therefore: $\mathbf{V = \Sigma V_i}$

Important

Gravitational field → Vector addition

Gravitational potential → Scalar addition

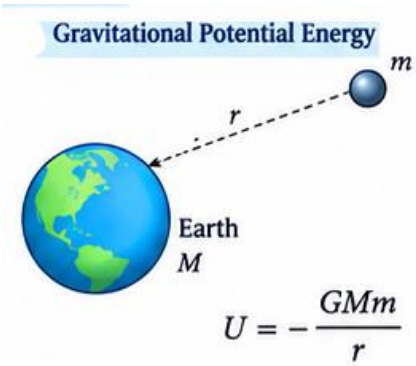
This distinction is extremely important.

4.11 Gravitational Potential Energy

Gravitational potential energy is the energy possessed by a mass because of its position in a gravitational field.

For a mass m at a distance r from a point mass M:

$$\mathbf{U = -GMm/r}$$



Where:

- U = gravitational potential energy
- M = source mass
- m = test mass
- r = separation between their centres

Since:

$$\mathbf{V = -GM/r}$$

We also have: $\mathbf{U = mV}$

4.12 Difference Between Gravitational Potential and Potential Energy

Gravitational Potential	Gravitational Potential Energy
Energy per unit mass	Total energy of the mass
V = U/m	U = mV
Unit: J kg^{-1}	Unit: J
Scalar quantity	Scalar quantity
Depends on source masses and position	Also depends on the test mass

Remember
Potential = Energy per unit mass
Potential energy = Mass × Potential

4.13 Gravitational Potential Energy of Two Masses

For two point masses M and m separated by r: **U = −GMm/r**
The zero of potential energy is taken at: **U = 0 at infinity**
Therefore, for a finite separation: **U < 0**
The negative value indicates that the system is gravitationally bound.

4.14 Work Done in Raising a Body Above Earth's Surface

This is an important derivation.
Consider a body of mass m initially at Earth's surface.
Let:

- M = mass of Earth
- R = radius of Earth
- h = height raised above Earth's surface

Initial distance from Earth's centre:

$$r_1 = R$$

Final distance:

$$r_2 = R+h$$

The gravitational potential energy at distance r is:

$$U = -\frac{GMm}{r}$$

Therefore, initial potential energy is:

$$U_1 = -\frac{GMm}{R}$$

Final potential energy is:

$$U_2 = -\frac{GMm}{(R+h)}$$

4.15 Change in Potential Energy

The change in gravitational potential energy is: $\Delta U = U_2 - U_1$
Therefore: $\Delta U = -\frac{GMm}{(R+h)} - \left(-\frac{GMm}{R}\right)$
Hence: $\Delta U = \frac{GMm}{R} - \frac{GMm}{(R+h)}$
Taking GMm common: $\Delta U = GMm\left[\frac{1}{R} - \frac{1}{(R+h)}\right]$
Taking the common denominator: $\Delta U = GMm\left[\frac{(R+h-R)}{(R(R+h))}\right]$
Therefore:

$$\boxed{\Delta U = \frac{GMmh}{R(R+h)}}$$

This is the exact increase in gravitational potential energy.

4.16 Work Done by External Agent

If the body is raised **slowly**, without any change in kinetic energy, the work done by the external agent is equal to the increase in potential energy.
Therefore:

$$\boxed{W_{ext} = \Delta U}$$

Hence:

$$\boldsymbol{W_{ext} = \frac{GMmh}{R(R+h)}}$$

Using: **g = GM/R²**
We can also write:

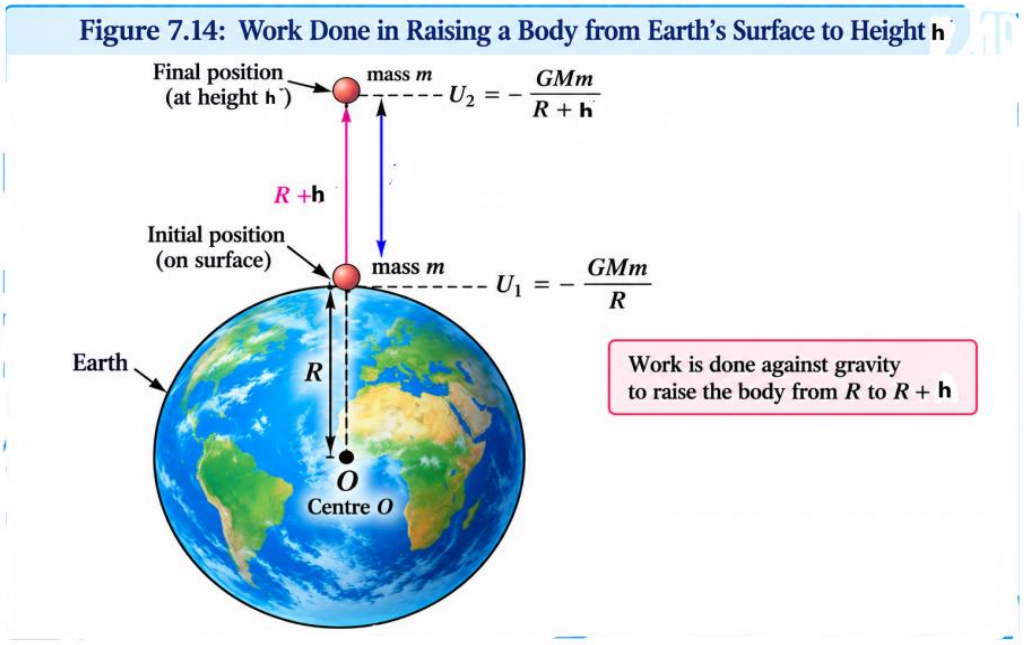
$$\boxed{W_{ext} = \frac{mgRh}{R+h}}$$

Or,

$$\boxed{W_{ext} = \frac{mgh}{1+\frac{h}{R}}}$$

This is the exact expression for the work required to raise a body from Earth's surface to height h, assuming Earth is spherical and neglecting rotational effects.
For h << R,

$$\boxed{\Delta U \approx mgh}$$



4.17 Small-Height Approximation

If: **h << R** then: **R+h ≈ R**
Therefore: **Wext = GMmh/[R(R+h)]** approximately becomes: **Wext ≈ GMmh/R²**
Since: **g = GM/R²**
We obtain:

$$\boxed{W_{ext} \approx mgh}$$

Thus, for small heights:

$$\boxed{\Delta U \approx mgh}$$

This is the familiar expression for gravitational potential energy near Earth's surface.

4.18 Important Comparison

Near Earth's surface: $\Delta U \approx mgh$

For any height h: $\Delta U = GMmh/[R(R+h)]$

The expression **mgh** is therefore an approximation valid when h is small compared with R.

4.19 Gravitational Binding Energy

A gravitationally bound system is one in which the components cannot be separated to infinity without supplying energy.

For two point masses M and m separated by r with gravitational potential energy as zero at infinity.

$$\boxed{U = -\frac{GMm}{r}}$$

The **binding energy** is the magnitude of their gravitational potential energy.

Therefore,

(A) The magnitude of gravitational potential energy between two masses is binding energy:

$$\boxed{E_{binding} = \frac{Gmm}{r}}$$

Thus: **E (binding) = −U** for a two-body gravitational system.

(B) For a satellite in circular orbit, the total mechanical energy is:

$$\boxed{E = -\frac{GMm}{2r}}$$

The energy required to remove it to infinity with zero final kinetic energy:

$$\boxed{E_{required} = \frac{Gmm}{2r}}$$

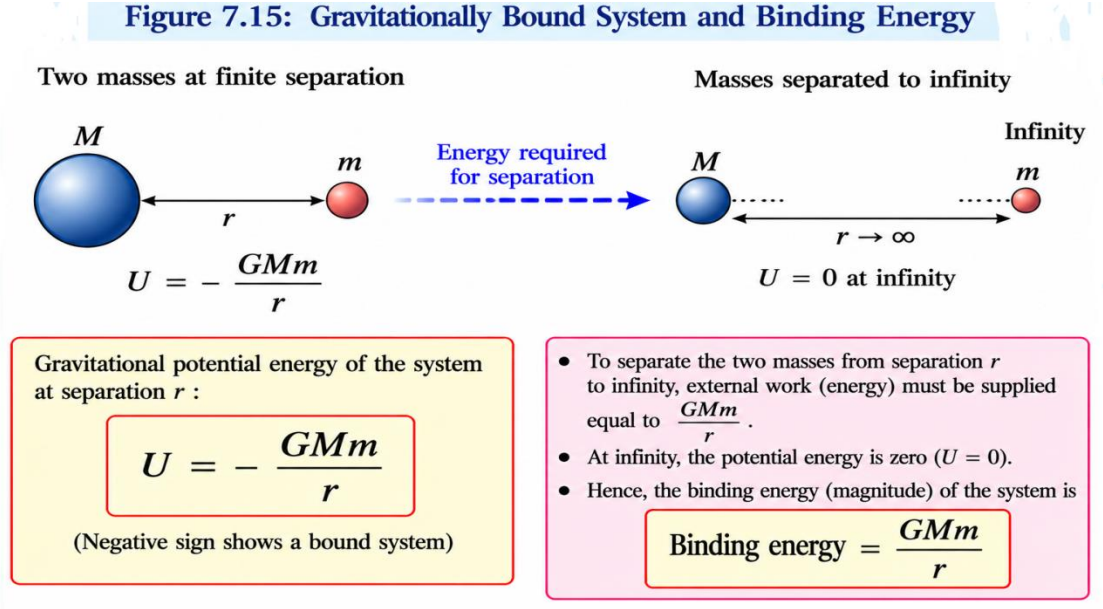
Important Point

Potential energy:

$$U = -\frac{GMm}{r}$$

Binding energy:

$$E_{binding} = \frac{Gmm}{r}$$



4.20 Gravitational Binding Energy of a Satellite

For a satellite of mass m orbiting a planet of mass M at orbital radius r, the total mechanical energy is negative.

The total energy is: **E = −GMm / (2r)**

Therefore, the magnitude:

|E| = GMm / (2r)

represents the energy required to remove the satellite from that circular orbit to infinity, if no further energy is lost.

This concept will be connected with satellite motion in Part 5.

4.21 Escape Velocity

Suppose a body is projected vertically upward from the surface of a planet.

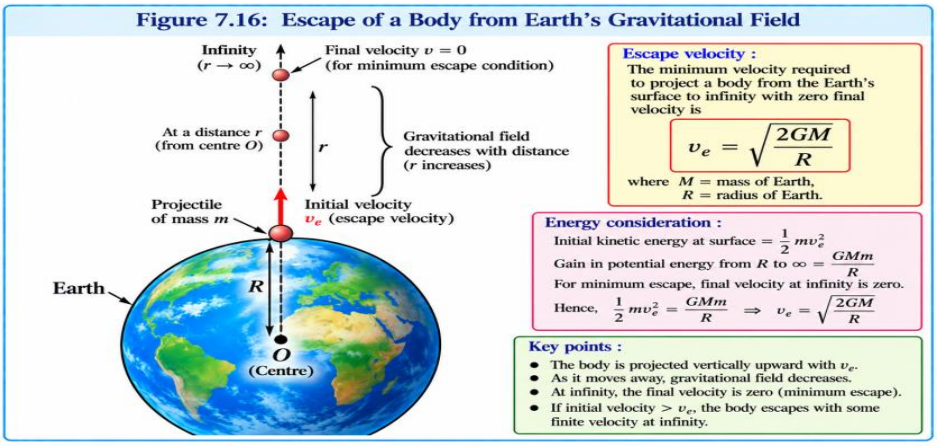
The minimum speed required so that the body can escape completely from the gravitational field of the planet and reach infinity with zero residual speed is called the **escape velocity**.

It is represented by: $\mathbf{v_e}$

4.22 Derivation of Escape Velocity

Consider:

- Mass of Earth = M
- Radius of Earth = R
- Mass of projectile = m
- Escape velocity = v_e



At Earth's surface, the total mechanical energy is:

Kinetic energy: K.E. = ½mv_e^2

Potential energy: P.E. = −GMm/R

Therefore: **E (initial) = ½mv_e^2 − GMm/R**

At infinity: **r → ∞**

Potential energy: **U∞ = 0**

For minimum escape velocity, final speed at infinity is zero.

Therefore: **K.E.∞ = 0**

Hence: **E (final) = 0**

By conservation of mechanical energy: **½mv_e^2 − GMm/R = 0**

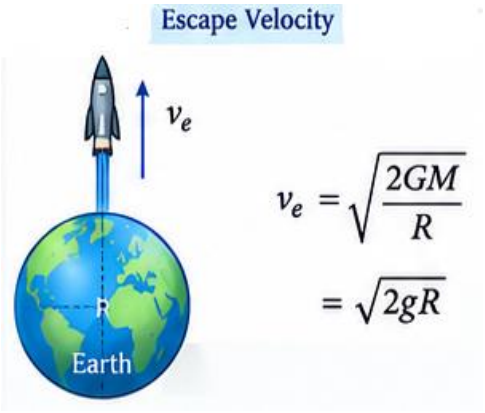
Therefore: **½mv_e^2 = GMm/R**

Cancel m: **½v_e^2 = GM/R**

Hence: **v_e^2 = 2GM/R**

Therefore:

$$v_e = \sqrt{\frac{2GM}{R}}$$



Using: **g = GM/R²**

We have: **GM/R = gR**

Therefore: $v_e = \sqrt{2gR}$

This is the escape velocity from the Earth's surface.

4.23 Important Properties of Escape Velocity

1. It does not depend on the mass of the projectile

From:

$$v_e = \sqrt{\frac{2GM}{R}}$$

The projectile of mass m does not appear.

Therefore, escape velocity is independent of the mass of the escaping body.

2. It depends on the planet

It depends on:

- Mass M of the planet
- Radius R of the planet

3. It is independent of direction for a spherically symmetric non-rotating body

At a given location, the minimum escape speed depends on the gravitational potential, not on the direction of projection.

4. Air resistance is neglected

The standard expression assumes no atmospheric resistance.

4.24 Relation Between Escape Velocity and Orbital Velocity

For a body near the surface:

Orbital velocity: $\mathbf{v_0 = \sqrt{(GM/R)}}$

Escape velocity: $\mathbf{v_e = \sqrt{(2GM/R)}}$

Therefore: $$\mathbf{v_e = \sqrt{2}\, v_0}$$

This is a very important relation.

4.25 Escape Velocity from a General Distance r

At a distance r from the centre of a spherical body:

$$\boxed{v_e = \sqrt{\frac{2GM}{r}}}$$

Therefore, escape velocity decreases as the distance from the centre increases.

4.26 Escape Velocity from Different Celestial Bodies

For a celestial body: $\mathbf{v_e = \sqrt{(2GM/R)}}$

Therefore: $\mathbf{v_e \propto \sqrt{(M/R)}}$

A body with greater mass and/or smaller radius generally has a larger escape velocity.

Examples of conceptual comparison

- A massive compact celestial body → large escape velocity.
- A less massive or larger-radius body → smaller escape velocity.

4.27 Example – Escape Velocity

If the orbital velocity close to the surface of a planet is: $\mathbf{v_0 = 8\ km\ s^{-1}}$

Then: $\mathbf{v_e = \sqrt{2}\, v_0}$

Therefore: $\mathbf{v_e \approx 1.414 \times 8 \approx 11.31\ km\ s^{-1}}$

4.28 Example – Escape Velocity at Height

At a distance: $\mathbf{r = 2R}$ from the centre of Earth: $\mathbf{v_e = \sqrt{(2GM/2R)}}$

Therefore: $\mathbf{v_e = \sqrt{(GM/R)}}$

Thus: $\mathbf{v_e\ (r = 2R) = v_0\ (surface)}$

This is a useful conceptual result.

4.29 Gravitational Potential and Escape

A body is gravitationally bound when its total mechanical energy is negative.
For escape:

$$E_{total} = 0$$

For an unbound object:

$$E_{total} > 0$$

Therefore:

Total energy	Nature of motion
E < 0	Bound
E = 0	Just escapes
E > 0	Unbound

This energy viewpoint is very useful in advanced problems.

4.30 Quick Revision Table

Concept	Important Result
Gravitational field	g = F/m
Field due to point mass	$g = GM/r^2$
Potential	V = W/m
Potential due to point mass	$V = -GM/r$
Potential energy	$U = -GMm/r$
Surface to height h	$\Delta U = GMmh/[R(R+h)]$
Small h	$\Delta U \approx mgh$
Binding energy	GMm/r
Escape velocity	$\sqrt{(2GM/R)}$
At distance r	$\sqrt{(2GM/r)}$
Orbital–escape relation	$v_e = \sqrt{2}\, v_0$
Bound system	E < 0
Just escape	E = 0
Unbound system	E > 0

Master Concept

Gravitational field describes the force-producing influence of mass, gravitational potential represents potential energy per unit mass, and gravitational potential energy determines whether a system is bound or unbound; escape velocity is the minimum speed required to make the total mechanical energy zero at infinity.

PART 5 – SATELLITE MOTION, ORBITAL ENERGY, GEOSTATIONARY SATELLITES AND WEIGHTLESSNESS

5.1 Learning Objectives

After studying this part, you should be able to:

- Explain the motion of an artificial satellite around Earth.
- Derive the expression for orbital velocity.
- Derive the time period of a satellite.
- Understand the dependence of orbital velocity and time period on orbital radius.
- Calculate the kinetic energy, potential energy and total energy of a satellite.
- Understand geostationary satellites and their applications.
- Explain the phenomenon of weightlessness in an orbiting satellite.
- Solve Board, NEET and JEE-level numerical problems based on satellite motion.

5.2 What Is a Satellite?

A **satellite** is a body that revolves around a planet or another celestial body under the influence of gravitational force.

Satellites may be: **Natural Satellites**

These occur naturally.

Example: The Moon is the natural satellite of Earth.

Artificial Satellites

These are launched by humans for various purposes.

Examples of applications:

- Communication
- Weather forecasting
- Remote sensing
- Navigation
- Scientific research
- Television and radio broadcasting
- Disaster monitoring

5.3 Why Does a Satellite Revolve Around Earth?

Consider a satellite of mass **m** revolving around Earth of mass **M** in a circular orbit of radius **r**.

The gravitational force between Earth and satellite is: $F = GMm/r^2$

This gravitational force acts towards the centre of Earth.

For circular motion, the required centripetal force is: $F_c = mv^2/r$

Therefore,

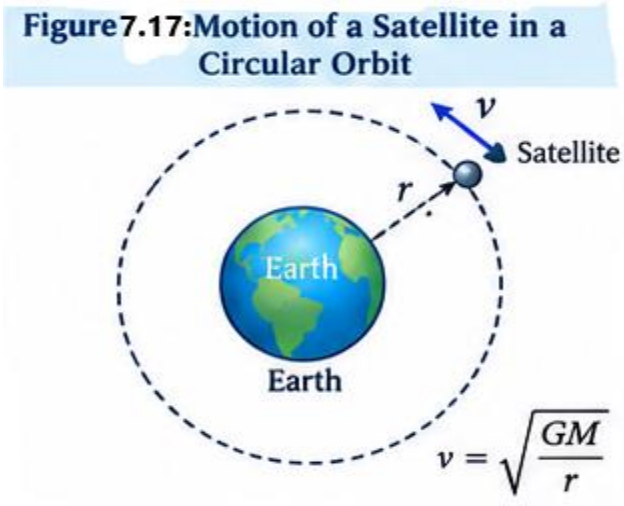
$$GMm/r^2 = mv^2/r$$

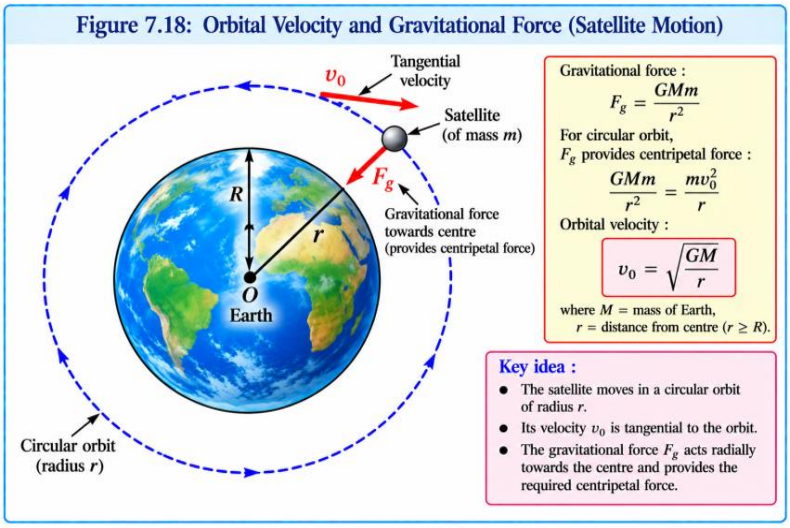
Cancelling m: $GM/r = v^2$

Hence,

Orbital Velocity: $v_0 = \sqrt{(GM/r)}$

This is the velocity required for a satellite to remain in a circular orbit of radius r.





5.4 Orbital Velocity at a Height h Above Earth

Let:

- R = radius of Earth
- h = height of satellite above Earth's surface

Then: r = R + h

Therefore:

$$v_0 = \sqrt{\frac{GM}{R+h}} = \sqrt{\frac{GM}{r}}$$

Using: $g = GM/R^2$

We get, for a satellite close to Earth's surface:

$$v_0 \approx \sqrt{gR}$$

For Earth: $v_0 \approx 7.9$ km s^{-1}

Thus, a satellite close to Earth's surface must move at approximately **7.9 km/s** to remain in a circular orbit, neglecting atmospheric resistance.

5.5 Important Dependence of Orbital Velocity

From: $v_0 = \sqrt{(GM/r)}$

We obtain: $\mathbf{v_0 \propto 1/\sqrt{r}}$

Therefore:

- If orbital radius increases, orbital velocity decreases.
- If orbital radius decreases, orbital velocity increases.

For two satellites around the same planet: $\mathbf{v_1/v_2 = \sqrt{(r_2/r_1)}}$

Important Result

A satellite in a **higher orbit moves more slowly** than a satellite in a lower circular orbit.

5.6 Angular Velocity of a Satellite

For circular motion: $v_0 = r\omega$

Therefore:

$$\omega = v_0/r$$

Substituting, we get:

$$v_0 = \sqrt{\frac{GM}{r}}$$

Hence:

$$\boxed{\omega = \sqrt{\frac{GM}{r^3}}}$$

Thus:

$$\omega \propto \frac{1}{r^{3/2}}$$

A satellite at a larger orbital radius has a smaller angular velocity.

5.7 Centripetal Acceleration of a Satellite

Centripetal acceleration is:

$$a = {v_0}^2/r$$

Since: $v_0^2 = GM/r$

Therefore,

$$a_c = \frac{GM}{r^2}$$

This is exactly equal to the gravitational acceleration at the satellite's orbital radius.

Hence:

$$\boldsymbol{a_c = g_r}$$

Where: $g_r = GM/r^2$

This shows that **gravity itself provides the centripetal acceleration required for orbital motion.**

5.8 Time Period of a Satellite

The time period T is the time required by the satellite to complete one revolution.

For circular motion,

$$T = 2\pi r/v_0$$

Using

$$v_0 = \sqrt{(GM/r)}$$

We get:

$$\boxed{T = \frac{2\pi r^{3/2}}{\sqrt{GM}}}$$

Therefore,

Time Period:

$$\boxed{T = 2\pi\sqrt{\frac{r^3}{GM}}}$$

Squaring:

$$T^2 = \frac{4\pi r^3}{GM}$$

Hence:

$$\boxed{T^2 \propto r^3}$$

This is the same mathematical form as **Kepler's Third Law**.

5.9 Time Period at Height h

Since: r = R + h,
We get: $\mathbf{T = 2\pi\sqrt{[(R + h)^3/GM]}}$
Thus, the time period increases as the height of the satellite increases.

5.10 Relation Between Orbital Velocity and Time Period

We know: $v_0 = 2\pi r/T$ and $v_0 = \sqrt{GM/r}$
Therefore:

$$\frac{2\pi r}{T} = \sqrt{\frac{GM}{r}}$$

This gives:

$$\boxed{T = 2\pi\sqrt{\frac{r^3}{GM}}}$$

This relation is useful in numerical problems involving orbital radius and time period.

5.11 Comparison of Two Satellites

For two satellites revolving around the same planet:

Orbital velocity: $\boldsymbol{v_1/v_2 = \sqrt{r_2/r_1}}$

Time period:

$$\frac{\boldsymbol{T_1}}{\boldsymbol{T_2}} = \left(\frac{\boldsymbol{r_1}}{\boldsymbol{r_2}}\right)^{\mathbf{3/2}}$$

Angular velocity:

$$\frac{\boldsymbol{\omega_1}}{\boldsymbol{\omega_2}} = \left(\frac{\boldsymbol{r_2}}{\boldsymbol{r_1}}\right)^{\mathbf{3/2}}$$

Important Shortcut

If $r_2 = 4r_1$, then: $\mathbf{v_2 = v_1/2}$ and $\mathbf{T_2 = 8T_1}$
This type of question is very common in NEET/JEE examinations.

5.12 Kinetic Energy of an Orbiting Satellite

The kinetic energy of a satellite is

$$K = \frac{1}{2}m{v_0}^2$$

Using,

$${v_0}^2 = \frac{GM}{r}$$

We get:

$$\boxed{K = \frac{GMm}{2r}}$$

Therefore,
The kinetic energy is positive.

5.13 Gravitational Potential Energy of a Satellite

The gravitational potential energy of a satellite at distance r from Earth's centre is:

$$\boxed{U = -\frac{GMm}{r}}$$

The negative sign indicates that the satellite is gravitationally bound to Earth.

5.14 Total Mechanical Energy of a Satellite

Total energy is: E = K + U
Substituting: $E = GMm/2r - GMm/r$
Therefore, **Total Energy:**

$$\boxed{E = -\frac{GMm}{2r}}$$

Hence: **E < 0** for a satellite in a stable circular orbit.
This negative value indicates that the satellite is in a **bound gravitational system**.

5.15 Important Energy Relations

For a satellite in a circular orbit:

Kinetic Energy: K = GMm/(2r)
Potential Energy: U = −GMm/r
Total Energy: E = −GMm/(2r)

Therefore:

$$\boxed{U = -2K}$$

And

$$\mathbf{E = -K}$$

Also,

$$E = U/2$$

These relations are extremely important for numerical problems.

Energy Summary Table

Quantity	Expression
Kinetic Energy	K = GMm/(2r)
Potential Energy	U = −GMm/r
Total Energy	E = −GMm/(2r)
Relation	U = −2K
Relation	E = −K
Relation	E = U/2

5.16 Energy Diagram of an Orbiting Satellite

As orbital radius r increases:

$K = GMm/2r$ decreases.

The magnitude of potential energy decreases:

$$|U| = GMm/r$$

The total energy approaches zero from the negative side:

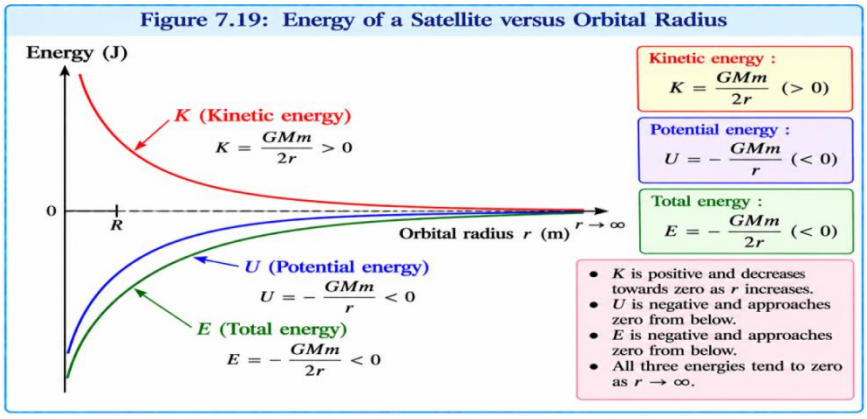
E = −GMm/(2r)

At infinity:

K → 0

U → 0

E → 0



5.17 Energy Required to Remove a Satellite from Its Orbit

For a satellite in a circular orbit: E = −GMm / (2r)

To take the satellite from its orbit to infinity with zero final kinetic energy:

Final energy = 0

Therefore, energy required is: ΔE = 0 − E

Hence,

ΔE = GMm / (2r)

This is equal to the **magnitude of the total energy** of the satellite.

5.18 Relation between Escape Velocity and Orbital Velocity

From Part 4:

$$v_e = \sqrt{\frac{2GM}{r}}$$

Orbital velocity is:

$$v_0 = \sqrt{\frac{GM}{r}}$$

Therefore,

$$v_e = \sqrt{2}v_0$$

Or

$$\frac{v_e}{v_0} = \sqrt{2}$$

For a satellite in a circular orbit:

- Orbital velocity = v_0
- Escape velocity at the same distance = $\sqrt{2}\, v_0$

Thus, the escape velocity is greater than the orbital velocity.

5.19 Geostationary Satellite

A **geostationary satellite** is an artificial satellite that appears stationary with respect to an observer on Earth.

For this to happen, the satellite must satisfy certain conditions.

Conditions for a Geostationary Satellite

1. Its orbit must be **circular**.
2. Its orbit must lie in the **equatorial plane**.
3. It must revolve from **west to east**, the same direction as Earth's rotation.
4. Its orbital period must be equal to Earth's rotational period.
5. Its angular velocity must be equal to Earth's angular velocity.

In school-level calculations, the period is taken as approximately: **T ≈ 24 h**

More precisely, the relevant period is one **sidereal day**, approximately 23 h 56 min.

5.20 Height of a Geostationary Satellite

For a geostationary satellite: $\mathbf{T = 2\pi\sqrt{(r^3/GM)}}$

Therefore,

$$r^3 = \frac{GMT^2}{4\pi^2}$$

Hence,

$$\boxed{r = \left(\frac{GMT^2}{4\pi^2}\right)^{3/2}}$$

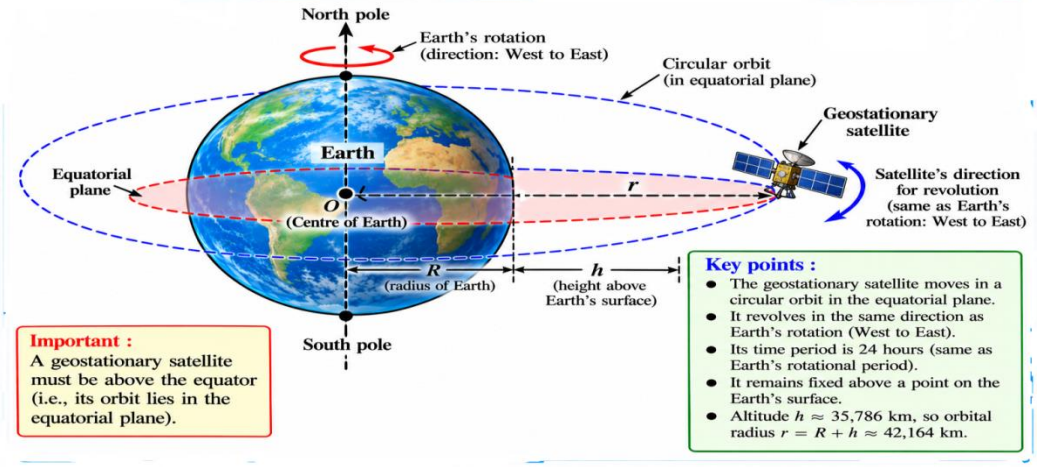
For Earth, the orbital radius is approximately: $\mathbf{r \approx 4.22 \times 10^7\ m}$

Since Earth's radius is approximately: $\mathbf{R \approx 6.4 \times 10^6\ m}$

The height above Earth's surface is: h = (r – R)

Therefore: $\mathbf{h \approx 3.58 \times 10^7\ m}$ or approximately: **h ≈ 35,800 km**

Figure 7.20: Geostationary Satellite



5.21 Why Does a Geostationary Satellite Appear Stationary?

Earth rotates about its axis.
The satellite also revolves around Earth in the same direction.
If: **Angular velocity of satellite = Angular velocity of Earth**
And both have the same period, the satellite remains above approximately the same point on Earth's equator.
Therefore, to an observer on Earth, it appears stationary.

5.22 Uses of Geostationary Satellites

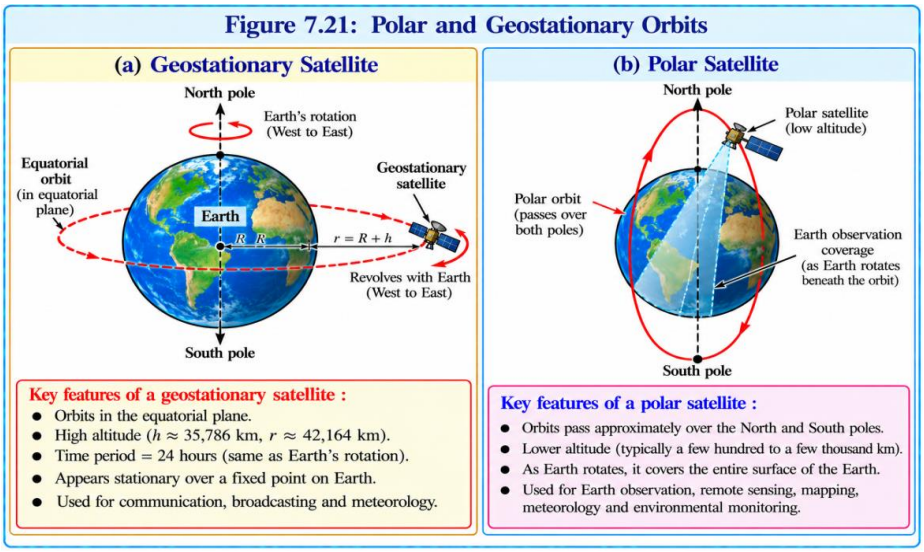
Geostationary satellites are particularly useful for:

- Television broadcasting
- Radio communication
- Telephone communication
- Internet and data communication
- Weather monitoring
- Satellite communication
- Continuous observation of a large region

Because the satellite remains apparently fixed relative to Earth, ground antennas can be directed toward a fixed position in the sky.

5.23 Polar Satellites

A **polar satellite** moves approximately over or near the Earth's polar regions.
Its orbit is generally at a much lower altitude than a geostationary satellite.
Polar satellites are widely used for:



- Earth observation
- Remote sensing
- Mapping
- Environmental monitoring
- Weather and climate studies

- Resource surveys

Unlike a geostationary satellite, a polar satellite does **not** remain fixed over one point on Earth's surface.

5.24 Weightlessness in a Satellite

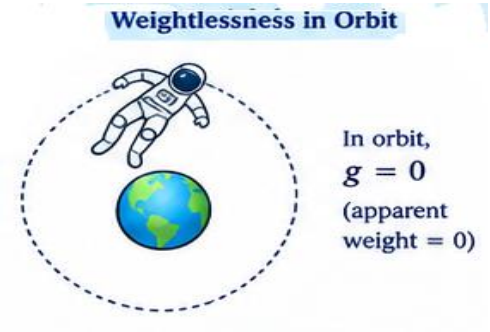
One of the most important applications of gravitation is the phenomenon of **weightlessness**.
Astronauts inside an orbiting spacecraft appear to have no weight.
This does **not** mean that Earth's gravitational force is zero.
In fact, gravity is the very force that keeps the spacecraft and astronauts in orbit.

5.25 Why Does Weightlessness Occur?

Consider an astronaut of mass m inside an orbiting spacecraft.
The gravitational force on the astronaut provides the required centripetal force: $\mathbf{GMm/r^2 = mv^2/r}$
The astronaut and spacecraft are both in **continuous free fall** towards Earth.
Because both fall together with the same acceleration, the astronaut does not press against the floor of the spacecraft.
Therefore, the normal reaction becomes: $\mathbf{N = 0}$
The apparent weight is: $\mathbf{W_{app} = N}$
Hence,

$$\mathbf{W_{app} = 0}$$

This condition is called **weightlessness**.



5.26 Weightlessness Does Not Mean Zero Gravity

This is a very important conceptual point.

Wrong statement:

"An astronaut is weightless because there is no gravity in space."

Correct statement:

An astronaut experiences apparent weightlessness because the astronaut and spacecraft are in **free fall together**.
Gravity is still acting.
At an orbital altitude, the value of g may be somewhat smaller than its surface value, but it is certainly not zero.

5.27 Apparent Weight and Actual Gravitational Force

For an astronaut in an orbiting spacecraft:
Gravitational force: $\mathbf{F_g = GMm/r^2}$
Apparent weight: $\mathbf{W_{app} = N = 0}$
Therefore: **Gravitational force ≠ apparent weight**
This distinction is very important for conceptual questions.

5.28 Solved Example 1 – Orbital Velocity

Question

Calculate the orbital velocity of a satellite close to Earth's surface.
Take: g = 9.8 m/s², R = 6.4 × 10⁶ m

Solution

For a near-Earth satellite: $v_0 = \sqrt{(gR)}$
Therefore: $v_0 = \sqrt{[(9.8)(6.4 \times 10^6)]}$ or $v_0 \approx 7.9 \times 10^3$ m/s
Hence: $v_0 \approx 79\ km\ s^{-1}$

Answer: Orbital velocity ≈ $7.9\ km\ s^{-1}$

5.29 Solved Example 2 – Comparing Two Satellites

Question

Two satellites revolve around the same planet in circular orbits of radii r and 4r. Compare their orbital velocities and time periods.

Solution

For orbital velocity: $v \propto 1/\sqrt{r}$
Given: $r_1 = r$ and $r_2 = 4r$
Therefore,

$$\frac{v_1}{v_2} = \sqrt{\frac{r_2}{r_1}} = \sqrt{\frac{4r}{r}} = 2$$

Hence,

$$v_1 : v_2 = 2 : 1$$

For time period: $T \propto r^{3/2}$
Therefore,

$$\frac{T_1}{T_2} = \left(\frac{r_1}{r_2}\right)^{3/2}$$

Or,

$$\frac{T_1}{T_2} = \left(\frac{r}{4r}\right)^{3/2} = \frac{1}{8}$$

Hence,

$$T_1 : T_2 = 1 : 8$$

Answer: $v_1 : v_2 = 2 : 1$ and $T_1 : T_2 = 1 : 8$

5.30 Solved Example 3 – Energy of a Satellite

Question

A satellite of mass m revolves around Earth in a circular orbit of radius r. Find its ratio of kinetic energy, potential energy and total energy.

Solution

Kinetic energy:

$$K = \frac{GMm}{2r} =$$

Potential energy:

$$U = -\frac{GMm}{r}$$

Total energy: E = K + U
Therefore,

$$E = -\frac{GMm}{2r}$$

Hence: **K: U: E = 1: −2: −1**

5.31 Solved Example 4 – Geostationary Satellite

Question

Why must a geostationary satellite be placed in the equatorial plane?

Answer

A geostationary satellite must remain above approximately the same point on Earth's surface.
For this to happen:

- Its orbital plane must coincide with Earth's equatorial plane.
- Its direction of revolution must be the same as Earth's rotation.
- Its period must equal Earth's rotational period.

Therefore, a satellite in an inclined or polar orbit cannot remain permanently above the same point on Earth's surface.

5.32 Solved Example 5 – Weightlessness

Question

An astronaut inside an orbiting spacecraft experiences zero apparent weight. Does this mean that Earth's gravitational force on the astronaut is zero?

Answer

No.
Earth's gravitational force is still acting on the astronaut.
The astronaut and spacecraft are both in free fall under Earth's gravitational field. Therefore, the astronaut does not exert a normal force on the spacecraft floor.
Thus: **N = 0** and the astronaut experiences **apparent weightlessness**.

5.33 Quick Revision – Part 5

Concept	Key Result
Orbital velocity	$v_0 = \sqrt{(GM/r)}$
Near-Earth orbital velocity	≈ 7.9 km/s
Angular velocity	$\omega = \sqrt{(GM/r^3)}$
Centripetal acceleration	GM/r^2
Time period	$T = 2\pi\sqrt{(r^3/GM)}$
Kepler's law	$T^2 \propto r^3$
Kinetic energy	$GMm/(2r)$
Potential energy	$-GMm/r$
Total energy	$-GMm/(2r)$
Escape velocity	$\sqrt{2}\, v_0$
Geostationary period	≈ 24 h
Geostationary orbit	Equatorial
Weightlessness	N = 0
Reason for weightlessness	Continuous free fall

5.34 Chapter 7 – Final Concept Summary

The complete logical chain of **Gravitation** can be remembered as:
Universal Gravitation → Gravitational Field → Acceleration due to Gravity (g) → Gravitational Potential → Gravitational Potential Energy → Escape Velocity → Orbital Motion → Orbital Velocity → Time Period → Satellite Energy → Geostationary Satellite → Weightlessness

Final Master Concept

The most important idea of satellite motion is:
Gravity provides the centripetal force required for a satellite to remain in orbit.
Therefore,
Gravitational Force = Centripetal Force
This leads directly to:

$$\boxed{v_0 = \sqrt{\frac{GM}{r}}}$$

And then to:

$$\boxed{T = 2\pi\sqrt{\frac{r^3}{GM}}}$$

And finally to the satellite-energy relations:

$$\boxed{K = \frac{GMm}{2r}; U = \frac{-GMm}{r}; E = \frac{-GMm}{2r}}$$