PART 1: Introduction to Oscillatory Motion, Periodic Motion and Simple Harmonic Motion (SHM)

1.1 Learning Objectives

After studying this part, students will be able to:

1. Understand periodic and oscillatory motion.
2. Differentiate between periodic motion and oscillatory motion.
3. Define mean position, extreme positions, amplitude, time period and frequency.
4. Understand the concept of Simple Harmonic Motion (SHM).
5. Identify the essential conditions for SHM.
6. Understand the basic equations describing SHM.
7. Recognise common examples of oscillatory motion in daily life and physics.

1.2 Introduction to Oscillatory Motion

Many objects in nature move repeatedly about a particular position. Such motion is observed in vibrating strings, pendulums, springs, tuning forks and several mechanical systems.

When an object moves repeatedly to and fro about a fixed equilibrium position, its motion is called **oscillatory motion**.

Examples of Oscillatory Motion

1. A child moving on a swing.
2. A mass attached to a spring and moving vertically.
3. The prongs of a vibrating tuning fork.
4. The oscillations of a simple pendulum through a small angle.
5. The vibrating diaphragm of a loudspeaker.
6. The vibrations of atoms and molecules about their equilibrium positions.

Important: Every oscillatory motion is periodic under ideal repeating conditions, but every periodic motion is not necessarily oscillatory.

For example, the uniform circular motion of a stone tied to a string is periodic, but the stone itself does not move to and fro along a fixed path.

1.3 Periodic Motion

A motion that repeats itself after equal intervals of time is called **periodic motion**.

The time after which the motion repeats itself is called its time period.

Examples

- Uniform circular motion.
- The revolution of Earth around the Sun.
- The rotation of Earth about its axis.
- The oscillation of a simple pendulum under ideal conditions.

Time Period (T)

The time taken by an object to complete one full cycle of periodic motion is called its **time period**.

The SI unit of time period is the second (s).

If an object completes N oscillations in time t, then

Time period: $T = t/N$

Where:

- T = time period
- t = total time taken
- N = number of complete oscillations

Example 1

A body completes 20 oscillations in 10 seconds. Calculate its time period.

Solution:

Given: N = 20, t = 10 s

Using,

$$T = t/N = 10/20 = \mathbf{0.5\ s}$$

Therefore, the time period is 0.5 s.

1.4 Oscillatory Motion

Oscillatory motion is a special type of motion in which a body moves repeatedly on either side of a fixed equilibrium position.

Consider a mass attached to a horizontal spring on a smooth surface.

When the mass is displaced from its equilibrium position and released, the spring exerts a restoring force that tends to bring the mass back towards equilibrium.

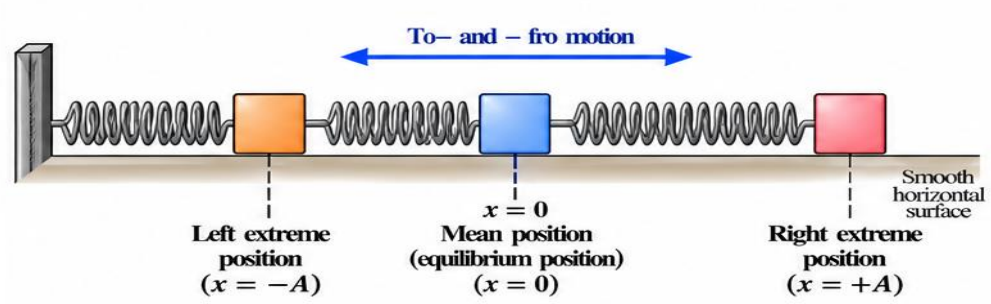
Due to inertia, the mass moves beyond the equilibrium position and continues its to-and-fro motion.

Under ideal conditions, the motion repeats periodically.

Important Features of Oscillatory Motion

1. The body moves on either side of an equilibrium position.
2. A restoring influence tends to bring the body towards equilibrium.
3. In ideal undamped oscillations, the motion continues with constant amplitude.
4. In real systems, energy losses may cause the amplitude to decrease with time.

Figure 13.1: Oscillatory Motion of a Mass Attached to a Spring



The block oscillates to and fro about the mean position between two extreme positions, executing simple harmonic motion.

1.5 Important Terms Used in Oscillatory Motion

1.5.1 Mean Position or Equilibrium Position

The position at which the net force on the oscillating body is zero is called its equilibrium position.

For a horizontal spring-mass system, the equilibrium position is the position where the spring is neither stretched nor compressed, provided no other horizontal force acts on the mass.

It is commonly represented by O.

1.5.2 Extreme Positions

The two positions farthest from the mean position, reached by the oscillating body, are called the extreme positions.
They are commonly represented by A and B.

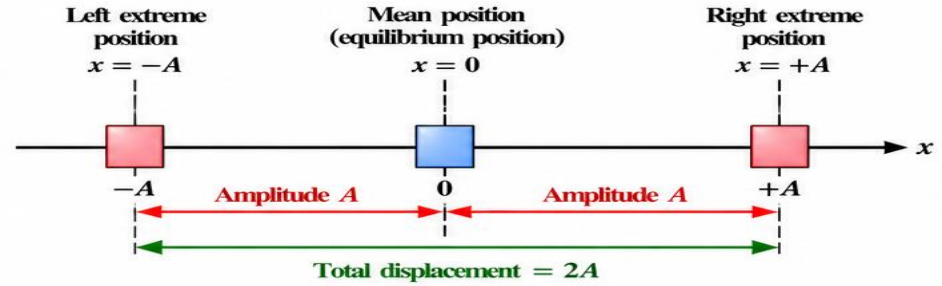
1.5.3 Amplitude (A)

The maximum displacement of an oscillating body from its mean position is called its **amplitude**.
If the mean position is O and the extreme positions are A and B, then
OA = OB = A
The SI unit of amplitude is the metre (m).
Important: Amplitude is the maximum magnitude of displacement, not the total distance travelled during one complete oscillation.

1.5.4 Complete Oscillation

One complete oscillation occurs when the body returns to its initial position with the same direction of motion.
For a body moving between extreme positions A and B through mean position O, one complete oscillation may be represented as: A → O → B → O → A

Figure 13.2: Mean Position, Extreme Positions and Amplitude



The maximum displacement from the mean position is called the amplitude A.
The block moves between $x = -A$ and $x = +A$.

1.5.5 Frequency (t)

The number of complete oscillations performed per unit time is called frequency.
If N oscillations occur in time t, then
Frequency: f = N/t
The SI unit of frequency is the hertz (Hz).
One hertz means one complete oscillation per second.

$$1 \text{ Hz} = 1 \text{ s}^{-1}$$

1.5.6 Relation Between Time Period and Frequency

Since the time period is the time taken for one oscillation, frequency and time period are reciprocals of each other.
Important Formula:

$$f = 1/T$$

or

$$T = 1/f$$

Where:

- f = frequency

- T = time period

Example 2

An oscillator completes 50 oscillations in 25 seconds. Calculate its frequency and time period.

Solution:

Given: N = 50, t = 25 s

Frequency: f = N/t = 50/25 = 2 Hz

Time period: T = 1/f = ½ **= 0.5 s**

Therefore: Frequency = 2 Hz

Time period = 0.5 s

1.6 Simple Harmonic Motion (SHM)

Simple Harmonic Motion is an important type of oscillatory motion.

Definition:

Simple Harmonic Motion is an oscillatory motion in which the restoring acceleration is directly proportional to the displacement from the equilibrium position and is always directed towards that position.

Mathematically,

$$a \propto -x$$

Therefore,

$$\mathbf{a = -\omega^2 x}$$

Where:

- a = acceleration of the particle
- x = displacement from the equilibrium position
- ω = angular frequency
- The negative sign indicates that acceleration is directed opposite to the displacement.

This is the fundamental condition for SHM.

Physical Meaning of the Negative Sign

When the particle is displaced towards the positive direction, its acceleration acts towards the negative direction.

Similarly, when the particle is displaced towards the negative direction, its acceleration acts towards the positive direction.

Thus, the acceleration always tends to bring the particle back towards the equilibrium position.

Important: The restoring acceleration is zero at the mean position and maximum in magnitude at the extreme positions.

1.7 Essential Conditions for Simple Harmonic Motion

For a particle to execute SHM, the following conditions must be satisfied:

1. There must be a stable equilibrium position.
2. A restoring force must act towards the equilibrium position when the particle is displaced.
3. The restoring force must be directly proportional to the displacement, within the range of motion.
4. The restoring force must act opposite to the displacement.

Since F = ma, the force condition for SHM is

$$\mathbf{F = -kx}$$

Where:

- F = restoring force
- k = positive constant
- x = displacement from equilibrium.

Using Newton's second law,

$$ma = -kx$$

Therefore,

$$a = -(k/m)x$$

Comparing this with

$$a = -\omega^2 x$$

We obtain

$$\boldsymbol{\omega^2 = k/m}$$

And

$$\boldsymbol{\omega = \sqrt{(k/m)}}$$

This relation applies to an ideal mass-spring system when k is the spring constant and m is the attached mass.

A more detailed discussion of the spring-mass system will be covered in a later part of this chapter.

Figure 13.3: Comparison of Periodic Motion and Oscillatory Motion

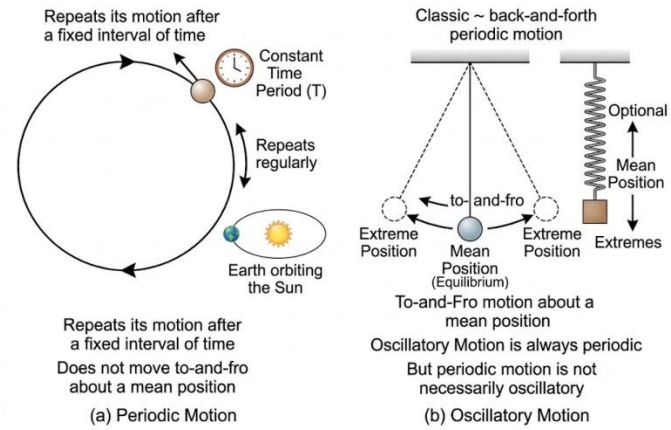
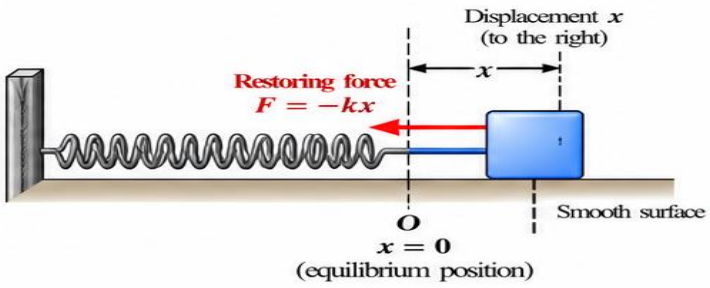


Figure 13.4: Restoring Force Directed Towards Equilibrium Position



The restoring force is always directed towards the mean position (equilibrium position) and is proportional to the displacement, $F = -kx$.

1.8 Displacement Equation of SHM

The displacement of a particle executing SHM changes continuously with time.

It can be expressed using a sine or cosine function. A general equation for SHM is

$$\boldsymbol{x = A \sin(\omega t + \varphi)}$$

Alternatively,

$$\boldsymbol{x = A \cos(\omega t + \varphi)}$$

Where:

- x = displacement at time t
- A = amplitude
- ω = angular frequency
- t = time
- φ = initial phase or phase constant.

The sine and cosine forms describe the same type of motion; the choice depends on the initial position and direction of motion.
The sine and cosine forms are mathematically equivalent, with an appropriate choice of phase constant.
For example, if the particle starts from the mean position and initially moves in the positive direction, its displacement can be written as

$$x = A \sin \omega t$$

If the particle starts from the positive extreme position, its displacement can be written as

$$x = A \cos \omega t$$

These equations will be derived and studied in detail in Part 2.

1.9 Angular Frequency (ω)

Angular frequency describes how rapidly the phase of an oscillation changes with time.
It is related to the time period and frequency by the following equations:

$$\omega = 2\pi/T$$

Since f = 1/T,

$$\omega = 2\pi f$$

where:

- ω = angular frequency, measured in rad s^{-1}
- T = time period, measured in seconds
- f = frequency, measured in hertz.

Although angular frequency is expressed in radians per second, the radian is dimensionless in SI dimensional analysis.

Example 3

An oscillator has a time period of 0.4 s. Calculate its frequency and angular frequency.
Solution:
Given: T = 0.4 s,
Frequency: f = 1/T = 1/0.4 = 2.5 Hz
Angular frequency: $\omega = 2\pi/T = 2\pi/0.4 = 5\pi$ rad s^{-1}
Therefore: **f = 2.5 Hz, ω ≈ 15.71 rad s^{-1}**

1.10 Difference Between Periodic Motion and Oscillatory Motion

Periodic Motion	Oscillatory Motion
The motion repeats itself after equal intervals of time.	The body moves repeatedly to and fro about an equilibrium position.
The motion need not occur about a fixed equilibrium position along a line.	The motion occurs about an equilibrium position.
Uniform circular motion is an example.	The motion of a spring-mass system is an example.
Periodic motion may or may not be oscillatory.	Ideal repeating oscillatory motion is periodic.

Exam Tip: Do not confuse uniform circular motion with oscillatory motion. Both may be periodic, but the actual motion of a body in uniform circular motion is not to and fro about an equilibrium position.

1.11 Important Conceptual Observations About SHM

1.11.1 At the Mean Position

At the mean position:

- Displacement, $x = 0$
- Restoring force, $F = 0$
- Acceleration, $a = 0$
- Speed is maximum in ideal SHM.

The speed is maximum because the potential energy associated with the oscillation is minimum and the kinetic energy is maximum.

1.11.2 At the Extreme Positions

At either extreme position:

- Displacement magnitude, $|x| = A$
- Speed, $v = 0$
- Acceleration magnitude, $|a| = \omega^2 A$
- Restoring force magnitude, $|F| = m\omega^2 A$.

The acceleration is directed towards the mean position.

1.11.3 Between the Mean and Extreme Positions

Between the mean and extreme positions:

- Displacement is non-zero.
- Acceleration acts towards the mean position.
- Speed changes continuously.
- Kinetic and potential energies change as the particle moves.

These relationships will be studied quantitatively in the following parts.

1.12 Solved Numerical Problems

Example 4: Calculating Time Period

A particle performs 120 oscillations in 60 seconds. Find its time period.

Solution:

Given: $N = 120$, $t = 60$ s

Time period: $T = t/N = 60/120$ **$= 0.5$ s**

Answer: 0.5 s.

Example 5: Calculating Angular Frequency

A particle executes SHM with a frequency of 5 Hz. Calculate its angular frequency.

Solution:

Given: $f = 5$ Hz

Using: $\omega = 2\pi f = 2\pi \times 5$ **$= 10\pi$ rad s^{-1}**

Therefore, $\omega \approx 31.42$ rad s^{-1}

Example 6: Maximum Acceleration in SHM

A particle executes SHM with amplitude 0.08 m and angular frequency 4 rad s^{-1}. Calculate its maximum acceleration.

Solution:

Given: $A = 0.08$ m, $\omega = 4$ rad s^{-1}

The acceleration in SHM is $a = -\omega^2 x$

The maximum magnitude of acceleration occurs when $|x| = A$.

Therefore: $a_{max} = \omega^2 A = 4^2 \times 0.08 = 16 \times 0.08$ **$= 1.28$ m s^{-2}**

Answer: 1.28 m s^{-2}.

PART 2: Displacement, Velocity, Acceleration, Phase and Graphs in Simple Harmonic Motion

2.1 Learning Objectives

After studying this part, students will be able to:

1. Write and interpret the displacement equation of SHM.
2. Derive the velocity equation for SHM.
3. Derive the acceleration equation for SHM.
4. Calculate maximum velocity and maximum acceleration.
5. Establish the relationship between velocity and displacement.
6. Understand phase and phase difference.
7. Interpret displacement-time, velocity-time and acceleration-time graphs.
8. Solve numerical problems based on SHM.
9. Apply these concepts to CBSE Board, NEET and JEE questions.

2.2 Displacement Equation of SHM

The displacement of a particle executing Simple Harmonic Motion changes periodically with time.

The general displacement equation is

$$\mathbf{x = A \sin(\omega t + \varphi)}$$

Alternatively,

$$\mathbf{x = A \cos(\omega t + \varphi)}$$

where:

- x = displacement at time t
- A = amplitude
- ω = angular frequency
- t = time
- φ = initial phase or phase constant.

The choice of sine or cosine depends on the initial conditions of motion.

2.2.1 SHM Starting from the Mean Position

Suppose a particle starts from the mean position at t = 0 and initially moves in the positive direction.

At t = 0: x = 0

Using the sine form,

$$x = A \sin \omega t$$

At t = 0,

$$x = A \sin 0 = 0$$

Thus, the equation satisfies the initial condition.

Therefore, for this choice of initial conditions,

$$\mathbf{x = A \sin \omega t}$$

2.2.2 SHM Starting from the Positive Extreme Position

Suppose the particle starts from the positive extreme position at t = 0.

At t = 0: x = A

Using the cosine form,

$$x = A \cos \omega t$$

At t = 0,

$$x = A \cos 0 = A$$

Therefore, the equation satisfies the initial condition.
Hence,

$$\mathbf{x = A \cos \omega t}$$

Important: Both equations describe SHM. The correct form depends on the initial position and direction of motion.

2.3 Velocity in SHM

Velocity is the rate of change of displacement with time.
Mathematically,

$$v = dx/dt$$

For a particle executing SHM, let its displacement be

$$x = A \sin \omega t$$

Differentiating with respect to time,

$$v = d(A \sin \omega t)/dt$$

Since A and ω are constants,

$$\mathbf{v = A\omega \cos \omega t}$$

This is the velocity equation for SHM when the particle starts from the mean position and moves initially in the positive direction.

2.3.1 Velocity in Terms of Displacement

We know that

$$x = A \sin \omega t$$

Therefore,

$$\sin \omega t = x/A$$

Using the trigonometric identity,

$$\sin^2\omega t + \cos^2\omega t = 1$$

We obtain

$$\cos^2\omega t = 1 - \sin^2\omega t$$

Substituting $\sin \omega t = x/A$,

$$\cos^2\omega t = 1 - x^2/A^2$$

Therefore,

$$\cos^2\omega t = (A^2 - x^2)/A^2$$

From the velocity equation,

$$v = A\omega \cos \omega t$$

Squaring both sides,

$$v^2 = A^2\omega^2 \cos^2\omega t$$

Substituting the value of $\cos^2\omega t$,

$$v^2 = A^2\omega^2 \times (A^2 - x^2)/A^2$$

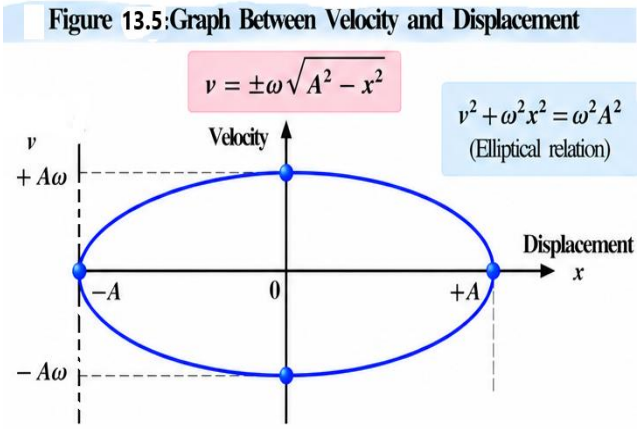
Hence,

$$\mathbf{v^2 = \omega^2(A^2 - x^2)}$$

Taking the square root,

$$\mathbf{v = \pm\omega\sqrt{(A^2 - x^2)}}$$

This is an important relation between velocity and displacement in SHM.
The positive or negative sign indicates the direction of motion.



Physical Interpretation

The equation shows that velocity depends on the displacement from the mean position.

- At the mean position, $x = 0$, velocity has maximum magnitude.
- At either extreme position, $x = \pm A$, velocity is zero.
- Between the mean and extreme positions, the speed decreases as the magnitude of displacement increases.

2.4 Maximum Velocity in SHM

The velocity equation is

$$v = A\omega \cos \omega t$$

The maximum magnitude of $\cos \omega t$ is 1.
Therefore,

$$|v|_{max} = A\omega$$

Hence, the **maximum speed** in SHM is

$$\mathbf{v_{max} = A\omega}$$

Since $\omega = 2\pi/T$,

$$\mathbf{v_{max} = 2\pi A/T}$$

Since $\omega = 2\pi f$,

$$\mathbf{v_{max} = 2\pi Af}$$

Important Conclusions

1. Maximum speed is directly proportional to amplitude.
2. Maximum speed is directly proportional to angular frequency.
3. Maximum speed occurs at the mean position.
4. Maximum speed is zero if the amplitude is zero.

Why Is Speed Maximum at the Mean Position?

At the mean position, the restoring force and acceleration are zero, but the particle has its maximum kinetic energy in ideal SHM.
The particle does not stop at the mean position because of inertia. It passes through the mean position with maximum speed.

2.5 Acceleration in SHM

Acceleration is the rate of change of velocity with time.
Mathematically,

$$a = dv/dt$$

We have derived the velocity equation:

$$v = A\omega \cos \omega t$$

Differentiating with respect to time,

$$a = d(A\omega \cos \omega t)/dt$$

Since A and ω are constants,

$$a = -A\omega^2 \sin \omega t$$

But

$$x = A \sin \omega t$$

Therefore,

$$\mathbf{a = -\omega^2 x}$$

This is the fundamental acceleration equation of SHM.

Meaning of the Negative Sign

The negative sign indicates that acceleration is always directed opposite to the displacement. Thus, acceleration always acts towards the equilibrium position.

- If x is positive, acceleration is negative.
- If x is negative, acceleration is positive.
- If x = 0, acceleration is zero.

2.6 Maximum Acceleration in SHM

The acceleration equation is

$$a = -\omega^2 x$$

The maximum magnitude of displacement is A.
Therefore, the maximum magnitude of acceleration occurs when |x| = A.
Hence,

$$\mathbf{a_{max} = \omega^2 A}$$

Using $\omega = 2\pi/T$,

$$\mathbf{a_{max} = 4\pi^2 A/T^2}$$

Using $\omega = 2\pi f$,

$$\mathbf{a_{max} = 4\pi^2 A f^2}$$

Important Conclusions

1. Maximum acceleration is directly proportional to amplitude.
2. Maximum acceleration is proportional to the square of angular frequency.
3. Maximum acceleration occurs at the extreme positions.
4. Acceleration is zero at the mean position.

2.7 Relationship Between Displacement, Velocity and Acceleration

For SHM starting from the mean position and moving initially in the positive direction:
Displacement: $x = A \sin \omega t$
Velocity: $v = A\omega \cos \omega t$
Acceleration: $a = -A\omega^2 \sin \omega t$
Since $x = A \sin \omega t$,

$$\mathbf{a = -\omega^2 x}$$

These equations describe the complete kinematics of SHM.

2.7.1 Comparison of Important Quantities

Quantity	Equation	Maximum Magnitude	Where Maximum?
Displacement	$x = A \sin \omega t$	A	Extreme positions
Velocity	$v = A\omega \cos \omega t$	$A\omega$	Mean position
Acceleration	$a = -\omega^2 x$	$A\omega^2$	Extreme positions

Phase Relationships

For $x = A \sin \omega t$:

- Velocity varies as $\cos \omega t$.
- Acceleration varies as $-\sin \omega t$.

Therefore:

1. Velocity and displacement differ in phase by $\pi/2$ rad, or 90°.
2. Acceleration and velocity differ in phase by $\pi/2$ rad, or 90°.

3. Acceleration and displacement differ in phase by π rad, or 180°.

Important: These phase relations apply to the sinusoidal quantities in ideal SHM. The signs depend on the chosen phase convention, but acceleration is always opposite to displacement.

2.8 Phase and Phase Difference

2.8.1 Phase

The phase of an oscillating particle specifies the stage of its oscillation at a particular instant.
For the equation

$$x = A \sin(\omega t + \varphi)$$

The phase is

$$\textbf{Phase} = \omega t + \varphi$$

Where:

- ωt = phase change with time
- φ = initial phase or phase constant.

Phase is expressed in radians.

2.8.2 Phase Difference

The difference between the phases of two oscillating quantities is called their phase difference.
Suppose two particles execute SHM with displacements

$$x_1 = A \sin \omega t$$

And

$$x_2 = A \sin(\omega t + \varphi)$$

Their phase difference is: $\Delta\varphi = \varphi$
when their angular frequencies are equal.

Important Phase Differences

Phase Difference	Meaning
0 rad or 0°	Same phase
π/2 rad or 90°	Difference of one-quarter cycle
π rad or 180°	Opposite phase
2π rad or 360°	One complete cycle; equivalent phase

Example

Two oscillations are represented by $x_1 = A \sin \omega t$ and $x_2 = A \sin(\omega t + \pi)$
Their phase difference is π radians.
Since: $\sin(\omega t + \pi) = -\sin \omega t$
We obtain: $x_2 = -x_1$
Therefore, the two oscillations are in opposite phase.

2.9 Graphs in SHM

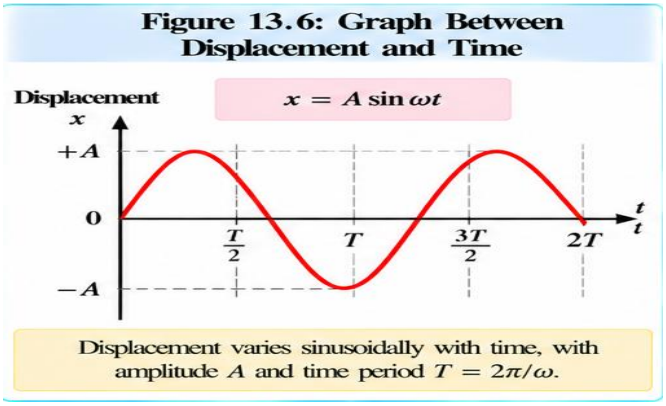
2.9.1 Displacement-Time Graph of SHM

For a particle starting from the mean position and moving initially in the positive direction, x = A sin ωt
The displacement varies sinusoidally with time.

Important Points

At t = 0: x = 0
At t = T/4: x = A
At t = T/2: x = 0
At t = 3T/4: x = −A

Figure 13.6: Graph Between Displacement and Time



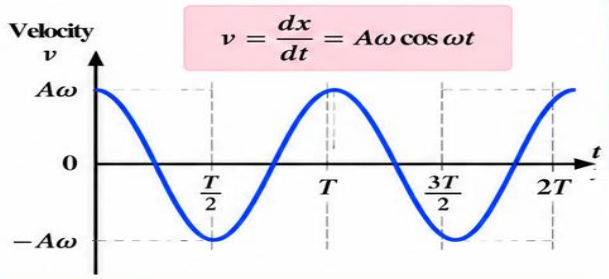
Displacement varies sinusoidally with time, with amplitude A and time period $T = 2\pi/\omega$.

At t = T: x = 0
Thus, the particle completes one full oscillation in time T.

2.9.2 Velocity-Time Graph of SHM

For the same motion,
v = Aω cos ωt
At t = 0: v = Aω
At t = T/4: v = 0
At t = T/2: v = −Aω
At t = 3T/4: v = 0
At t = T: v = Aω
Therefore, the velocity varies periodically with time.

Figure 13.7: Graph Between Velocity and Time

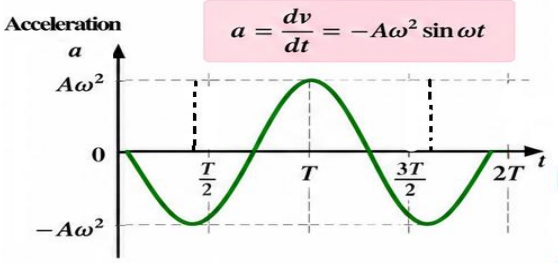


Velocity is a cosine function of time with amplitude $A\omega$.

2.9.3 Acceleration-Time Graph of SHM

The acceleration equation is
a = −Aω² sin ωt
At t = 0: a = 0
At t = T/4: a = −Aω²
At t = T/2: a = 0
At t = 3T/4: a = +Aω²
At t = T: a = 0
Therefore, acceleration also varies sinusoidally with time.

Figure 13.8: Graph Between Acceleration and Time

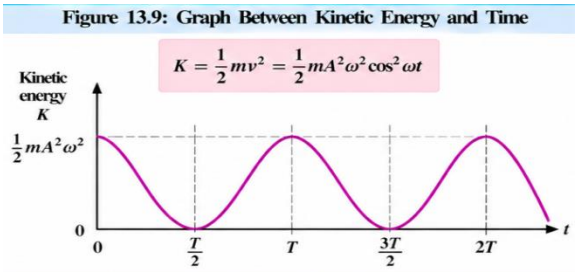


Acceleration is a sine function of time with amplitude $A\omega^2$ and is opposite in phase to displacement.

2.9.4 Kinetic Energy –Time Graph of SHM

Since, $K = (1/2)mv^2 = (1/2)mA^2\omega^2cos^2\omega t$

Figure 13.9: Graph Between Kinetic Energy and Time

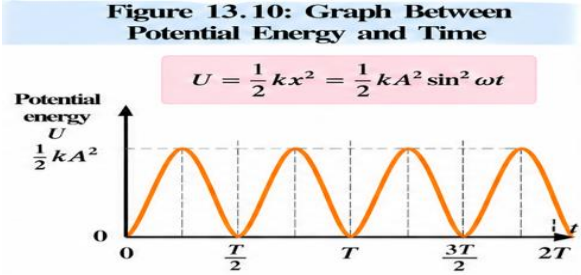


Kinetic energy varies as $\cos^2\omega t$ with period $T/2$.
At $t = 0$, $K = \frac{1}{2}mA^2\omega^2$ (maximum).

2.9.5 Potential Energy-Time Graph of SHM

Since, $U = (1/2)kx^2 = (1/2)mA^2\omega^2sin^2\omega t$
As, $k = m\omega^2$

Figure 13.10: Graph Between Potential Energy and Time

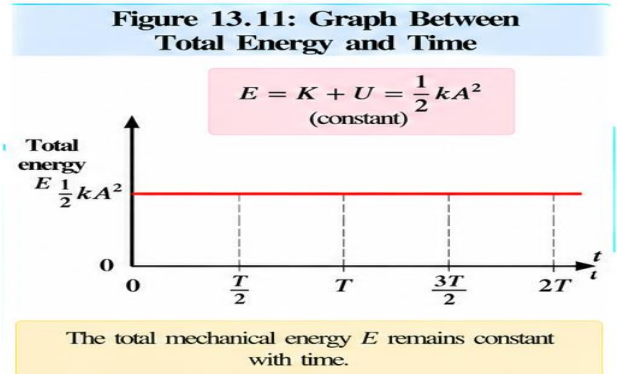


Potential energy varies as $\sin^2\omega t$ with period $T/2$ and maximum value $\frac{1}{2}kA^2$.

2.9.6 Total Energy versus Time Graph

Since, $E = K + U = (1/2)mA^2\omega^2cos^2\omega t + (1/2)mA^2\omega^2sin^2\omega t$

Or, $E = (1/2)mA^2\omega^2 = (1/2)kA^2$



2.12 Comparison of Displacement, Velocity and Acceleration Graphs

For the SHM equation: x = A sin ωt

the three quantities vary as follows:

Time	Displacement x	Velocity v	Acceleration a
0	0	+Aω	0
T/4	+A	0	−Aω²
T/2	0	−Aω	0
3T/4	−A	0	+Aω²
T	0	+Aω	0

Important Observations

1. Displacement and acceleration are always opposite in sign.
2. Velocity leads displacement by π/2 rad in the chosen sine convention.
3. Acceleration and displacement differ in phase by π rad.
4. The time period of displacement, velocity and acceleration is the same: T.
5. The maximum magnitudes are A, Aω and Aω², respectively.

2.13 Numerical Problems

Example 1: Maximum Speed

A particle executes SHM with amplitude 0.05 m and time period 2 s. Calculate its maximum speed.

Solution

Given: A = 0.05 m, T = 2 s

Maximum speed: vmax = 2πA/T

Substituting,

$$v_{max} = 2\pi \times 0.05/2 = 0.05\pi$$

Therefore: **vmax ≈ 0.157 m s⁻¹**

Example 2: Maximum Acceleration

A particle executes SHM with amplitude 0.10 m and frequency 2 Hz. Calculate its maximum acceleration.

Solution
Given: A = 0.10 m, f = 2 Hz
Angular frequency:

$$\omega = 2\pi f = 2\pi \times 2 = 4\pi \text{ rad s}^{-1}$$

Maximum acceleration:
$a_{max} = \omega^2 A = (4\pi)^2 \times 0.10 = 16\pi^2 \times 0.10 = 1.6\pi^2 \approx$ **15.79 m s⁻²**

Example 3: Speed at a Given Displacement

A particle executes SHM with amplitude 0.10 m and angular frequency 5 rad s⁻¹. Calculate its speed when its displacement from the mean position is 0.06 m.

Solution
Given: A = 0.10 m, ω = 5 rad s⁻¹, x = 0.06 m
Using: $v = \pm\omega\sqrt{(A^2 - x^2)}$
The speed is the magnitude of velocity:
$|v| = \omega\sqrt{(A^2 - x^2)} = 5\sqrt{[(0.10)^2 - (0.06)^2]} = 5\sqrt{(0.0100 - 0.0036)} = 5\sqrt{0.0064} = 5 \times 0.08$
Therefore: **Speed = 0.40 m s⁻¹**
The direction of velocity cannot be determined from the displacement alone.

Example 4: Acceleration at a Given Displacement

A particle executes SHM with angular frequency 10 rad s⁻¹. Find its acceleration when its displacement is +0.03 m.

Solution
Given: ω = 10 rad s⁻¹, x = +0.03 m
Using: $a = -\omega^2 x = -(10)^2 \times 0.03 = -100 \times 0.03 =$ **−3.0 m s⁻²**
The negative sign indicates that acceleration acts towards the mean position.

Example 5: Finding the Time Period from Maximum Speed

A particle executing SHM has amplitude 0.20 m and maximum speed 2.0 m s⁻¹. Calculate its time period.

Solution
Given: A = 0.20 m, vmax = 2.0 m s⁻¹
Using: $v_{max} = 2\pi A/T$
Rearranging: $T = 2\pi A/v_{max}$
Substituting,

$$T = 2\pi \times 0.20/2.0 = 0.20\pi$$

Therefore: **T ≈ 0.628 s**

PART 3: Restoring Force, Spring-Mass System and Energy in Simple Harmonic Motion

3.1 Learning Objectives

After studying this part, students will be able to:

1. Explain restoring force and Hooke's law.
2. Derive the equation of motion of a spring-mass system.
3. Derive the angular frequency and time period of a spring-mass system.
4. Understand the effect of mass and spring constant on the time period.
5. Derive the expressions for kinetic and potential energy in SHM.
6. Establish the conservation of total mechanical energy in ideal SHM.
7. Understand the variation of kinetic energy and potential energy with displacement.
8. Solve numerical problems based on spring-mass systems and energy in SHM.
9. Apply these concepts to CBSE Board, NEET and JEE questions.

3.2 Restoring Force

A restoring force is a force that tends to bring a displaced body back towards its equilibrium position.

When a body is displaced from its stable equilibrium position, a restoring force may act towards the equilibrium position.

Examples include:

- The force exerted by a stretched or compressed spring.
- The restoring force acting on a simple pendulum displaced through a small angle.
- The restoring force acting on a vibrating tuning fork.
- The restoring force associated with small vibrations of atoms in a solid.

For SHM, the restoring force must be directly proportional to the displacement and opposite in direction.

Therefore: $\mathbf{F \propto -x}$

Or,

$$\mathbf{F = -kx}$$

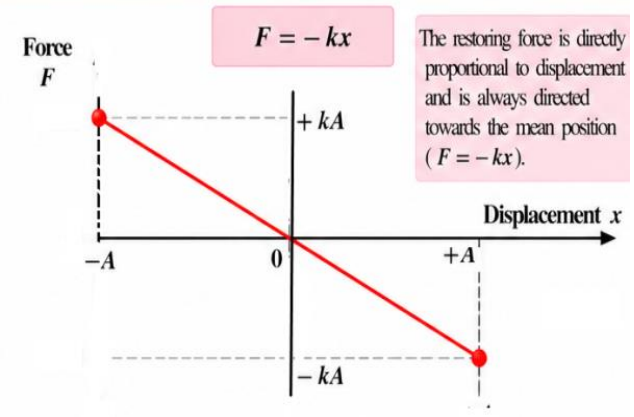
Where:

- F = restoring force
- k = positive force constant
- x = displacement from the equilibrium position.

The negative sign indicates that the force acts opposite to the displacement.

Important: A restoring force does not automatically guarantee SHM. For SHM, the restoring force must be proportional to displacement over the range of motion.

Figure 13.12: Graph Between Force and Displacement



3.3 Hooke's Law

Hooke's law states that, within the elastic limit, the restoring force exerted by an ideal spring is directly proportional to its extension or compression.

Mathematically,

$$\mathbf{F = -kx}$$

The magnitude of the restoring force is

$$|\mathbf{F}| = \mathbf{k}|\mathbf{x}|$$

Where k is the spring constant.

Spring Constant (k)

The spring constant represents the stiffness of a spring.
From Hooke's law,

$$k = |F|/|x|$$

Its SI unit is newton per metre (N m^{-1}).

Physical Meaning

- A larger spring constant means a stiffer spring.
- A smaller spring constant means a softer spring.
- For the same displacement, a stiffer spring exerts a greater restoring force.

Example

A spring has a spring constant of 200 N m^{-1}. It is stretched by 0.05 m. Calculate the magnitude of the restoring force.

Solution:

Given: k = 200 N m^{-1}, x = 0.05 m
Using: |F| = k|x| = 200 × 0.05 **= 10 N**
The restoring force acts opposite to the direction of the displacement.

3.4 Spring-Mass System

Consider a block of mass m attached to one end of an ideal horizontal spring. The other end of the spring is fixed to a rigid wall.
Assume that:

1. The surface is smooth and frictionless.
2. The spring obeys Hooke's law.
3. The mass of the spring is negligible compared with the attached mass.
4. Air resistance is neglected.

When the block is displaced by a distance x from its equilibrium position, the spring exerts a restoring force.

According to Hooke's law: $F = -kx$
Using Newton's second law: $F = ma$
Therefore: $ma = -kx$
Dividing by m, we get:

$$\mathbf{a = -(k/m)x}$$

This equation has the same form as the acceleration equation of SHM:

$$a = -\omega^2 x$$

Comparing the two equations,

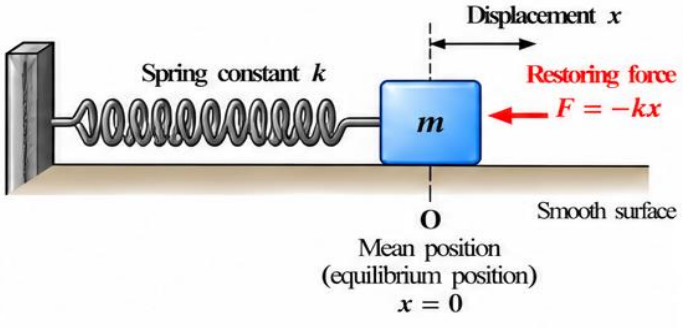
$$\omega^2 = k/m$$

Therefore,

$$\boldsymbol{\omega = \sqrt{(k/m)}}$$

Hence, the motion of the block is SHM.

Figure 13.13 : Horizontal Spring–Mass System and Restoring Force



A mass attached to a horizontal spring executes simple harmonic motion. The restoring force $F = -kx$ is directed towards the mean position (equilibrium position).

3.5 Time Period of a Spring-Mass System

We have established that the angular frequency of an ideal spring-mass system is

$$\omega = \sqrt{(k/m)}$$

For SHM,

$$\omega = 2\pi/T$$

Therefore,

$$2\pi/T = \sqrt{(k/m)}$$

Rearranging,

$$T = 2\pi\sqrt{(m/k)}$$

Hence, the time period of a mass attached to an ideal spring is

$$\mathbf{T = 2\pi\sqrt{(m/k)}}$$

This is one of the most important formulas in the chapter.

Frequency of a Spring-Mass System

Since f = 1/T,

$$\mathbf{f = (1/2\pi)\sqrt{(k/m)}}$$

Important Conclusions

From

$$T = 2\pi\sqrt{(m/k)}$$

We conclude that:

1. The time period increases with the square root of mass.
2. The time period decreases with the square root of spring constant.
3. A heavier mass attached to the same spring oscillates more slowly.
4. A stiffer spring with the same mass produces a shorter time period.
5. For an ideal spring obeying Hooke's law, the time period is independent of amplitude.

Important: The independence from amplitude applies to the ideal linear spring model. Real springs may depart from SHM at sufficiently large extensions or compressions.

3.6 Effect of Mass and Spring Constant on Time Period

The time period is: $T = 2\pi\sqrt{}$ (m/k)

3.6.1 If the Mass Is Increased

Suppose the mass becomes four times its original value.
Then,

$$T' = 2\pi\sqrt{(4m/k)}$$
$$T' = 2 \times 2\pi\sqrt{(m/k)}$$

Therefore,

$$\mathbf{T' = 2T}$$

Thus, quadrupling the mass doubles the time period.

3.6.2 If the Spring Constant Is Increased

Suppose the spring constant becomes four times its original value.
Then,

$$T' = 2\pi\sqrt{(m/4k)}$$

Therefore,

$$\mathbf{T' = T/2}$$

Thus, quadrupling the spring constant halves the time period.

3.6.3 General Ratio Formula

For two spring-mass systems,

$$T_1 = 2\pi\sqrt{(m_1/k_1)} \text{ and } T_2 = 2\pi\sqrt{(m_2/k_2)}$$

Dividing,

$$\mathbf{T_2/T_1 = \sqrt{[(m_2k_1)/(m_1k_2)]}}$$

This formula is useful for solving comparative numerical problems.

3.7 Spring-Mass System in the Vertical Direction

Consider a mass m suspended from a vertical spring.

When the mass is attached, the spring stretches until the upward spring force balances the downward gravitational force.

At the new equilibrium position,

$$ke = mg$$

where e is the static extension of the spring.

Therefore,

$$\mathbf{e = mg/k}$$

Now displace the mass by a small distance x downward from this equilibrium position.

The spring's extension becomes e + x.

The upward spring force is

$$F_s = -k(e + x)$$

Taking downward as positive, gravity acts downward with force +mg.

Therefore, the net force is

$$F = mg - k(e + x)$$

Expanding,

$$F = mg - ke - kx$$

Since ke = mg,

$$F = mg - mg - kx$$

Hence,

$$\mathbf{F = -kx}$$

Using Newton's second law,

$$ma = -kx$$

Therefore,

$$a = -(k/m)x$$

Thus, the motion is SHM, with

$$\mathbf{\omega = \sqrt{(k/m)}}$$

And

$$\mathbf{T = 2\pi\sqrt{(m/k)}}$$

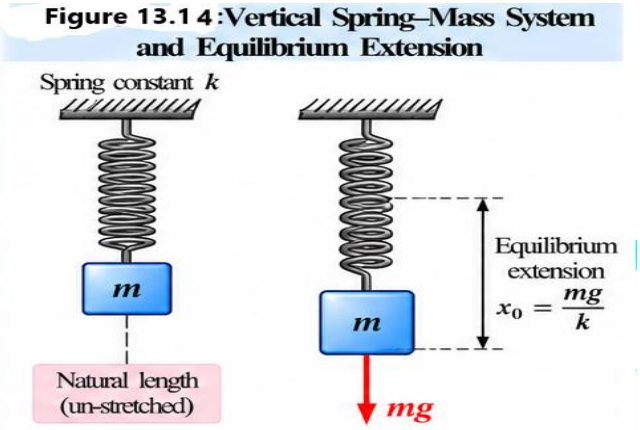


Figure 13.14 :Vertical Spring–Mass System and Equilibrium Extension

In the vertical position, the spring extends by $x_0 = \frac{mg}{k}$ due to the weight of the mass. Small oscillations about this equilibrium position are simple harmonic.

Important Conclusion

For an ideal vertical spring-mass system, gravity changes the equilibrium position but does not change the time period of small oscillations about that equilibrium position.

3.8 Energy in Simple Harmonic Motion

In ideal SHM, energy continuously changes between kinetic energy and potential energy.

At the mean position, the particle has maximum speed and maximum kinetic energy.

At the extreme positions, the particle is momentarily at rest and has maximum potential energy.

In the absence of friction or other dissipative forces, the total mechanical energy remains constant.
The total mechanical energy is: **E = K + U**
Where:

- E = total mechanical energy
- K = kinetic energy
- U = potential energy.

For an ideal spring-mass oscillator, the potential energy is the elastic potential energy of the spring, measured relative to the equilibrium position.

3.9 Potential Energy in SHM

Consider an ideal spring obeying Hooke's law.
The restoring force is: $F = -kx$
The external force required to stretch the spring slowly from x = 0 to x = x must balance the restoring force.
Its magnitude at an intermediate displacement y is: $F_{ext} = ky$
The small work done by the external force is

$$dW = F_{ext}\, dy$$

Therefore,

$$dW = ky\, dy$$

The work required to stretch the spring from 0 to x is

$$W = \int_0^x ky\, dy = \left(k\frac{y^2}{2}\right)_0^x = \frac{1}{2}kx^2$$

This work is stored as elastic potential energy.
Therefore: **$U = \frac{1}{2}kx^2$**
For a spring-mass oscillator, $\omega^2 = k/m$, so $k = m\omega^2$.
Substituting: **$U = \frac{1}{2}m\omega^2x^2$**
This is the potential energy expression for an ideal spring-mass oscillator, taking the equilibrium position as the zero of potential energy.

Important Observations

1. Potential energy is zero at x = 0 under this choice of reference.
2. Potential energy is always non-negative for an ideal spring.
3. Potential energy increases with the square of displacement.
4. Potential energy is maximum at the extreme positions.

At $x = \pm A$, $U_{max} = \frac{1}{2}kA^2$
Therefore: **$U_{max} = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2A^2$**

3.10 Kinetic Energy in SHM

The kinetic energy of a particle of mass m moving with speed v is: $K = \frac{1}{2}mv^2$
From Part 2, we know that

$$v^2 = \omega^2(A^2 - x^2)$$

Substituting into the kinetic energy equation,

$$K = \frac{1}{2}m\omega^2(A^2 - x^2)$$

Therefore,

$$\mathbf{K = \frac{1}{2}m\omega^2(A^2 - x^2)}$$

Since $k = m\omega^2$,

$$\mathbf{K = \frac{1}{2}k(A^2 - x^2)}$$

Expanding,

$$\mathbf{K = \frac{1}{2}kA^2 - \frac{1}{2}kx^2}$$

Since the maximum potential energy is

$$U_{max} = \frac{1}{2}kA^2$$

We can also write

$$\mathbf{K = \frac{1}{2}kA^2 - U}$$

Important Observations

1. Kinetic energy is maximum at the mean position.
2. Kinetic energy is zero at the extreme positions.
3. Kinetic energy decreases as the magnitude of displacement increases.
4. Kinetic energy is always non-negative.

At the mean position, $x = 0$: $K = \frac{1}{2}m\omega^2A^2$

Therefore: $\mathbf{K_{max} = \frac{1}{2}m\omega^2A^2 = \frac{1}{2}kA^2}$

At the extreme positions, $x = \pm A$: $K = 0$

3.11 Total Mechanical Energy in SHM

The total mechanical energy is the sum of kinetic and potential energies.

$$\boxed{E = K + U}$$

We have,

$$K = \frac{1}{2}m\omega^2(A^2 - x^2)$$

And

$$U = \frac{1}{2}m\omega^2x^2$$

Adding,

$$E = \frac{1}{2}m\omega^2(A^2 - x^2) + \frac{1}{2}m\omega^2x^2$$

Expanding,

$$E = \frac{1}{2}m\omega^2A^2 - \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2x^2$$

The displacement-dependent terms cancel.

Therefore,

$$\mathbf{E = \frac{1}{2}m\omega^2A^2}$$

Since $k = m\omega^2$,

$$\mathbf{E = \frac{1}{2}kA^2}$$

Hence, the total mechanical energy of an ideal harmonic oscillator is

$$\mathbf{E = \frac{1}{2}m\omega^2A^2 = \frac{1}{2}kA^2}$$

Important Conclusions

1. Total mechanical energy remains constant in ideal SHM.
2. Total energy is proportional to the square of amplitude.
3. Doubling the amplitude makes the total energy four times as large, provided the oscillator's mass and angular frequency remain unchanged.
4. Kinetic energy and potential energy continuously interchange.
5. The total mechanical energy is independent of the instantaneous displacement.

Note: These energy relations assume ideal SHM without energy loss due to friction, air resistance or other dissipative effects.

3.12 Energy at Different Positions

For an ideal spring-mass oscillator,

$$U = \tfrac{1}{2}k\,x^2$$
$$K = \tfrac{1}{2}k(A^2 - x^2)$$
$$E = \tfrac{1}{2}kA^2$$

The energy distribution at important positions is shown below.

Position	Displacement	Kinetic Energy	Potential Energy	Total Energy
Positive extreme	$x = +A$	0	$\frac{1}{2}kA^2$	$\frac{1}{2}kA^2$
Mean position	$x = 0$	$\frac{1}{2}kA^2$	0	$\frac{1}{2}kA^2$
Negative extreme	$x = -A$	0	$\frac{1}{2}kA^2$	$\frac{1}{2}kA^2$
Intermediate position	$x = \pm A/2$	$3kA^2/8$	$kA^2/8$	$\frac{1}{2}kA^2$

Verification at $x = \pm A/2$

Potential energy: $U = \frac{1}{2}k(A/2)^2 = kA^2/8$
Kinetic energy: $K = \frac{1}{2}k[A^2 - A^2/4] = \frac{1}{2}k(3A^2/4) = 3kA^2/8$
Total energy: $E = 3kA^2/8 + kA^2/8 = 4kA^2/8 = \mathbf{\frac{1}{2}kA^2}$
Thus, the total mechanical energy remains constant.

3.13 Graphs of Energy in SHM

3.13.1 Potential Energy versus Displacement

The potential energy is: $U = \frac{1}{2}kx^2$
This is a quadratic equation in x.
Therefore, the potential-energy graph is an upward-opening parabola.
Important points:

- $U = 0$ at $x = 0$.
- U increases with x^2.
- $U = \frac{1}{2}kA^2$ at $x = \pm A$.

3.13.2 Kinetic Energy Versus Displacement

The kinetic energy is: $K = \frac{1}{2}k(A^2 - x^2)$
This is a downward-opening parabola over the allowed interval $-A \le x \le +A$.
Important points:

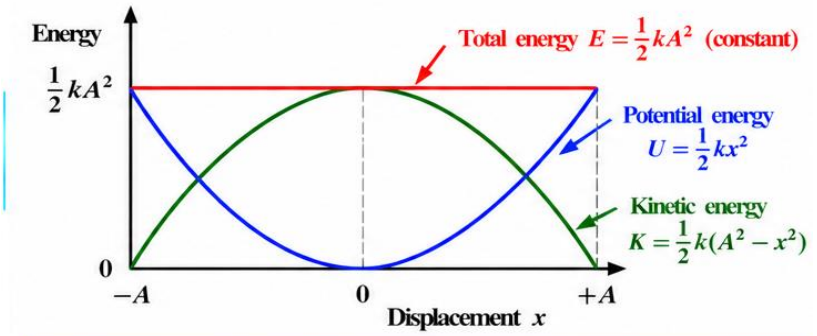
- K is maximum at $x = 0$.
- $K = 0$ at $x = \pm A$.
- K decreases symmetrically on either side of the mean position.

Suggested diagram for the book: Draw displacement x along the horizontal axis and kinetic energy K along the vertical axis. Plot $K = \frac{1}{2}k(A^2 - x^2)$. The graph reaches its maximum value $\frac{1}{2}kA^2$ at $x = 0$ and meets the horizontal axis at $x = -A$ and $x = +A$.

3.13.3 Total Energy versus Displacement

The total mechanical energy is: $E = \frac{1}{2}kA^2$
It does not depend on x.
Therefore, the total-energy graph is a horizontal straight line.

Figure 13.15:Graph Between Mechanical Energy and Displacement



The total mechanical energy E remains constant.
Kinetic energy is maximum at the mean position ($x = 0$) and zero at $x = \pm A$,
while potential energy is zero at $x = 0$ and maximum at $x = \pm A$.

3.14 Numerical Problems

Example 1: Time Period of a Spring-Mass System

A mass of 0.5 kg is attached to a spring of spring constant 200 N m⁻¹. Calculate its time period.

Solution

Given: m = 0.5 kg, k = 200 N m⁻¹

Using,

$$T = 2\pi\sqrt{(m/k)}$$

Substituting,

$$T = 2\pi\sqrt{(0.5/200)} = 2\pi\sqrt{0.0025} = 2\pi \times 0.05$$

Therefore: **T = 0.1π s or T ≈ 0.314 s**

Example 2: Maximum Kinetic Energy

A mass of 0.2 kg executes SHM with amplitude 0.1 m and angular frequency 10 rad s⁻¹. Calculate its maximum kinetic energy.

Solution

Given: m = 0.2 kg, A = 0.1 m, ω = 10 rad s⁻¹

Maximum kinetic energy is

$$K_{max} = \tfrac{1}{2}m\omega^2A^2$$

Substituting,

$$K = \tfrac{1}{2} \times 0.2 \times 10^2 \times 0.1^2 = 0.1 \times 100 \times 0.01$$

Therefore, **K = 0.1 J**

Example 3: Potential Energy at a Given Displacement

A spring with spring constant 400 N m⁻¹ is displaced by 0.05 m from its equilibrium position. Calculate the elastic potential energy stored in the spring.

Solution

Given: k = 400 N m⁻¹, x = 0.05 m

Potential energy:

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2} \times 400 \times (0.05)^2 = 200 \times 0.0025$$

Therefore, **U = 0.5 J**

Example 4: Kinetic and Potential Energy

A particle executes SHM with amplitude 0.2 m and spring constant 100 N m^{-1}. Calculate its kinetic energy, potential energy and total energy when its displacement is 0.1 m.

Solution

Given: A = 0.2 m, x = 0.1 m, k = 100 N m^{-1}

Step 1: Total Energy

$$E = \tfrac{1}{2}kA^2 = \tfrac{1}{2} \times 100 \times (0.2)^2 = 50 \times 0.04 = \mathbf{2\ J}$$

Step 2: Potential Energy

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2} \times 100 \times (0.1)^2 = 50 \times 0.01 = \mathbf{0.5\ J}$$

Step 3: Kinetic Energy

$$K = E - U = 2 - 0.5 = \mathbf{1.5\ J}$$

Therefore:

- Kinetic energy = 1.5 J
- Potential energy = 0.5 J
- Total energy = 2 J

Example 5: Effect of Increasing Mass

A mass attached to an ideal spring has a time period of 2 s. What will be its new time period if the mass is increased to four times its original value without changing the spring?

Solution

The time period is

$$T = 2\pi\sqrt{(m/k)}$$

For the new mass, $m' = 4m$

Therefore,

$$T' = 2\pi\sqrt{(4m/k)} = 2T$$

Since T = 2 s, therefore, $\mathbf{T' = 4\ s}$

PART 4: Simple Pendulum, Time Period and Combination of Springs

4.1 Learning Objectives

After studying this part, students will be able to:

1. Define a simple pendulum and identify its components.
2. Explain the forces acting on a bob of a simple pendulum.
3. Derive the equation of motion of a simple pendulum.
4. Derive the time period of a simple pendulum for small oscillations.
5. Understand the role of the small-angle approximation.
6. Explain the factors affecting the time period of a simple pendulum.
7. Understand the behaviour of a pendulum in different gravitational fields.
8. Derive the effective spring constant for springs connected in series and parallel.
9. Solve numerical problems based on simple pendulums and combinations of springs.

4.2 Simple Pendulum

A simple pendulum consists of a small, heavy bob suspended from a fixed support by a light, inextensible string.

For an ideal simple pendulum, we assume that:

1. The string has negligible mass.
2. The string does not stretch.
3. The bob is treated as a point mass.
4. The support is fixed.
5. Air resistance is neglected.
6. The oscillations are small.

When the bob is displaced from its equilibrium position and released, it oscillates to and fro in a vertical plane.

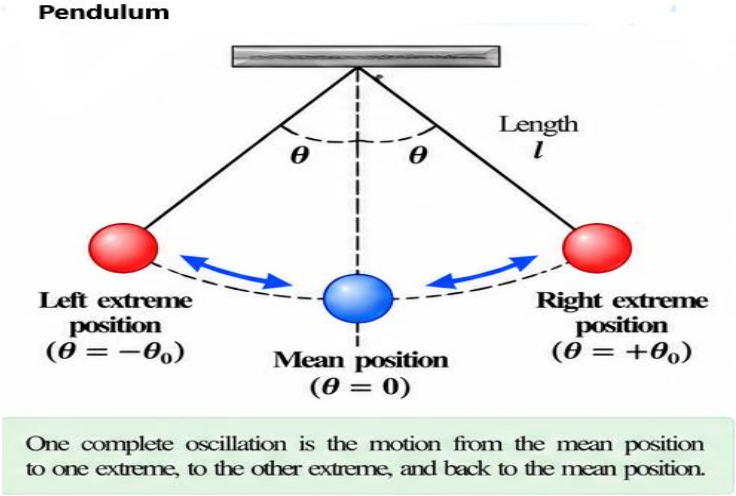
For small angular displacements, its motion is approximately Simple Harmonic Motion.

Important Components

- **Point of suspension:** The fixed point from which the string hangs.
- **Length of pendulum (L):** The distance from the point of suspension to the centre of the bob.
- **Bob:** The small mass attached to the string.
- **Mean position:** The lowest equilibrium position of the bob.
- **Angular displacement (θ):** The angle made by the string with the vertical.
- **Extreme positions:** The maximum angular positions reached during oscillation.

.

Figure 13.16: One Complete Oscillation of a Simple Pendulum



One complete oscillation is the motion from the mean position to one extreme, to the other extreme, and back to the mean position.

4.3 Forces Acting on a Simple Pendulum

Consider a simple pendulum of bob mass m and string length L.
Let the bob be displaced through an angle θ from the vertical.
Two forces act on the bob:

1. Gravitational force mg, acting vertically downward.
2. Tension T, acting along the string towards the point of suspension.

The weight mg can be resolved into two components:

- Radial component: mg cos θ
- Tangential component: mg sin θ

The radial component acts along the string, while the tangential component acts along the arc of motion.
The tangential component provides the restoring force.
Taking displacement towards the positive angular direction as positive, the restoring force is

$$\mathbf{F = -mg \sin \theta}$$

The negative sign indicates that the force acts towards the equilibrium position.
Using Newton's second law along the tangential direction,

$$ma_t = -mg \sin \theta$$

Since tangential acceleration is

$$a_t = L\alpha$$

Where α is the angular acceleration,

$$mL\alpha = -mg \sin \theta$$

Dividing by mL,

$$\boldsymbol{\alpha = -(g/L) \sin \theta}$$

This is the exact angular equation of motion for an ideal simple pendulum.
However, because sin θ is not exactly proportional to θ at large angles, the exact equation does not describe SHM for arbitrary angular displacements.
For small oscillations, we use the small-angle approximation.

4.4 Small-Angle Approximation

For sufficiently small angles measured in radians,

$$\mathbf{\sin \theta \approx \theta}$$

Similarly,

$$\tan\theta \approx \theta$$

And

$$\cos\theta \approx 1$$

These approximations are useful when θ is small.
For example, when θ = 0.1 rad, sin θ is approximately 0.0998, which is very close to θ = 0.1.
Using: sin θ ≈ θ in the angular equation,

$$\alpha = -(g/L)\sin\theta$$

We obtain

$$\boldsymbol{\alpha \approx -(g/L)\theta}$$

This equation shows that the angular acceleration is directly proportional to the angular displacement and opposite in direction.
Therefore, for small oscillations, the motion of a simple pendulum is approximately SHM.
Important: The small-angle approximation requires θ to be measured in radians.

4.5 Derivation of the Time Period of a Simple Pendulum

Consider a simple pendulum of length L and bob mass m.
Let its angular displacement from the vertical be θ.
The restoring torque about the point of suspension is

$$\tau = -mgL\sin\theta$$

For small oscillations,

$$\sin\theta \approx \theta$$

Therefore,

$$\tau \approx -mgL\theta$$

Using the rotational equation of motion,

$$\tau = I\alpha$$

For a point mass m at distance L from the axis of rotation,

$$I = mL^2$$

Therefore,

$$mL^2\alpha = -mgL\theta$$

Dividing by mL^2,

$$\alpha = -(g/L)\theta$$

Since $\alpha = d^2\theta/dt^2$,

$$\boldsymbol{d^2\theta/dt^2 = -(g/L)\theta}$$

Rearranging,

$$\boldsymbol{d^2\theta/dt^2 + (g/L)\theta = 0}$$

The standard differential equation of SHM is

$$d^2x/dt^2 + \omega^2 x = 0$$

Comparing the two equations,

$$\omega^2 = g/L$$

Therefore,

$$\boldsymbol{\omega = \sqrt{(g/L)}}$$

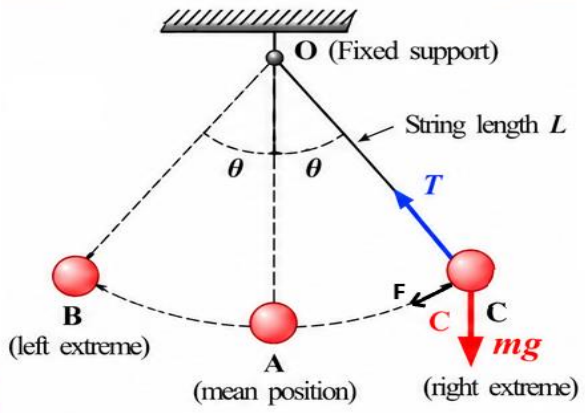
We know that

$$\omega = 2\pi/T$$

Hence,

$$2\pi/T = \sqrt{(g/L)}$$

Figure 13.17: Simple Pendulum Showing Its Mean and Extreme Positions



A simple pendulum oscillates about the mean position *A* between two extreme positions *B* and *C*. The restoring force is the component of weight *mg* along the tangential direction.

Rearranging,

$$\mathbf{T = 2\pi\sqrt{(L/g)}}$$

This is the time-period formula for a simple pendulum undergoing small oscillations.

Final Result

Time period: $T = 2\pi\sqrt{(L/g)}$
Frequency: $f = 1/T$
Therefore: $\mathbf{f = (1/2\pi)\sqrt{(g/L)}}$
Angular frequency: $\boldsymbol{\omega = \sqrt{(g/L)}}$

Important Observation

The time period of a simple pendulum depends on its length and the local acceleration due to gravity.
For small oscillations, it is independent of the mass of the bob.

4.6 Factors Affecting the Time Period of a Simple Pendulum

The time period is: $T = 2\pi\sqrt{(L/g)}$

4.6.1 Effect of Length

The time period is proportional to the square root of length: $\mathbf{T \propto \sqrt{L}}$
If the length becomes four times its original value,

$$T' = 2\pi\sqrt{(4L/g)}$$

Therefore,

$$\mathbf{T' = 2T}$$

Thus, quadrupling the length doubles the time period.
If the length becomes nine times its original value, the time period becomes three times its original value.

4.6.2 Effect of Gravitational Acceleration

The time period is inversely proportional to the square root of gravitational acceleration.

$$\mathbf{T \propto 1/\sqrt{g}}$$

If the gravitational acceleration becomes four times its original value,

$$T' = 2\pi\sqrt{(L/4g)}$$

Therefore,

$$\mathbf{T' = T/2}$$

A stronger gravitational field produces a shorter time period.

4.6.3 Effect of Mass of the Bob

The mass does not appear in the time-period formula.
Therefore, for an ideal simple pendulum undergoing small oscillations, the time period is independent of the mass of the bob.

4.6.4 Effect of Amplitude

For small oscillations, the time period is approximately independent of amplitude.
At larger amplitudes, the motion is not exactly SHM, and the time period becomes slightly greater than the small-angle value.
Exam Tip: When using $T = 2\pi\sqrt{(L/g)}$, check that the question assumes small oscillations unless a more advanced treatment is explicitly required.

4.7 Simple Pendulum in Different Gravitational Fields

The time period of a simple pendulum is

$$T = 2\pi\sqrt{(L/g)}$$

Thus, the time period depends on the local gravitational acceleration.

4.7.1 On Earth

Taking $g \approx 9.8\ m\ s^{-2}$, $T = 2\pi\sqrt{(L/9.8)}$

For a pendulum of length 1 m,

$$T = 2\pi\sqrt{(1/9.8)}\ \mathbf{T \approx 2.01\ s}$$

4.7.2 On the Moon

The gravitational acceleration on the Moon is approximately one-sixth of that on Earth.

Thus,

$$g_{moon} = g_{earth}/6$$

The lunar time period is

$$T_{moon} = 2\pi\sqrt{(L/g_{moon})}$$

Substituting $g_{moon} = g_{earth}/6$,

Therefore,

$$\mathbf{T_{moon} = \sqrt{6}\ T_{earth}}$$

Hence, for the same pendulum length, the time period on the Moon is approximately 2.45 times its time period on Earth.

4.7.3 In a Freely Falling Lift

In a freely falling lift, the effective gravitational acceleration relative to the lift is approximately zero.

Therefore, the usual restoring effect of gravity on a simple pendulum disappears.

The bob does not execute ordinary gravitational pendulum oscillations relative to the lift.

Important: This conclusion assumes ideal free fall and neglects small tidal effects.

4.8 Seconds Pendulum

A seconds pendulum is a pendulum whose time period is 2 seconds.

Therefore: $T = 2\ s$

Using: $T = 2\pi\sqrt{(L/g)}$

Substituting $T = 2\ s$,

$$2 = 2\pi\sqrt{(L/g)}$$

Dividing by 2,

$$1 = \pi\sqrt{(L/g)}$$

Squaring,

$$1 = \pi^2 L/g$$

Therefore,

$$\mathbf{L = g/\pi^2}$$

Using $g = 9.8\ m\ s^{-2}$,

$$L = 9.8/\pi^2$$

Hence,

$$\mathbf{L \approx 0.993\ m}$$

Thus, the length of a seconds pendulum is approximately 0.993 m under standard near-Earth gravitational conditions.

Note: A second pendulum takes one second to move from one extreme to the other and another second to return to the original extreme. Its complete time period is 2 seconds.

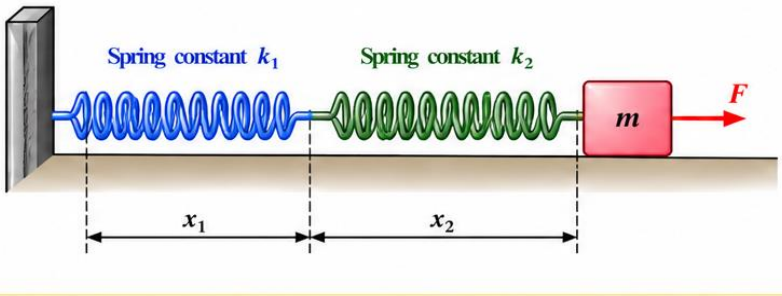
4.9 Combination of Springs

Springs can be connected in series or parallel to obtain a desired effective spring constant.
The effective spring constant determines the restoring force produced by the combination for a given displacement.
Let k_1 and k_2 be the spring constants of two ideal springs.

4.9.1 Springs Connected in Series

When two springs are connected in series, the same force acts through each spring in static equilibrium.
Let their extensions be x_1 and x_2.
The total extension is: $x = x_1 + x_2$
For the first spring: $F = k_1x_1$
Therefore: $x_1 = F/k_1$
For the second spring: $F = k_2x_2$
Therefore: $x_2 = F/k_2$
Adding the extensions: $x = F/k_1 + F/k_2$
Therefore: $x = F(1/k_1 + 1/k_2)$
For the equivalent spring constant keq,
$F = keq\ x$
Thus: $x = F/keq$
Comparing,
$F/keq = F(1/k_1 + 1/k_2)$
Cancelling F,
$1/keq = 1/k_1 + 1/k_2$
Therefore,
$$\mathbf{1/keq = 1/k_1 + 1/k_2}$$
For two springs,
$$\mathbf{keq = k_1k_2/(k_1 + k_2)}$$

Figure 13.18: Two Springs Connected in Series



Two springs of spring constants k_1 and k_2 are connected in series. The same force F acts in both springs. The total extension is $x = x_1 + x_2$.

Important Conclusion

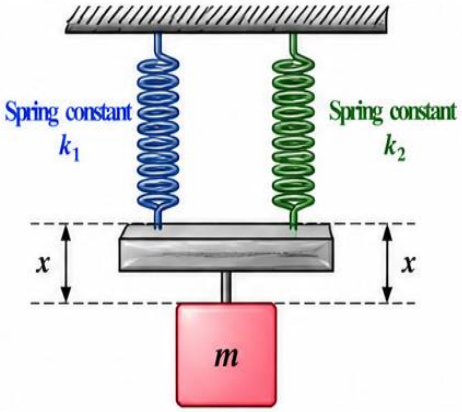
For two springs connected in series, the effective spring constant is smaller than either individual spring constant.
For n springs connected in series,
$$\mathbf{1/keq = 1/k_1 + 1/k_2 + ... + 1/k_n}$$

4.9.2 Springs Connected in Parallel

When two springs are connected in parallel, their ends are attached to common supports.
If the mass is displaced by x, both springs undergo the same displacement.
The restoring forces are:
$$F_1 = -k_1x$$
And
$$F_2 = -k_2x$$
The total restoring force is: $F = F_1 + F_2$
Therefore: $F = -k_1x - k_2x\ F = -(k_1 + k_2)x$
Comparing with $F = -keq\ x$,

We obtain: **keq = $k_1 + k_2$**
For n springs connected in parallel:
keq = $k_1 + k_2 + ... + k_n$

Figure 13.19: Two Springs Connected in Parallel



Two springs of spring constants k_1 and k_2 are connected in parallel.
Both springs have the same extension x, which is equal to the displacement of the mass.

Important Conclusion

For two springs connected in parallel, the effective spring constant is greater than either individual spring constant.

4.10 Time Period of a Mass Attached to Combined Springs

For a mass m attached to an effective spring constant keq, the time period is

$T = 2\pi\sqrt{(m/keq)}$

4.10.1 Two Springs in Series

For two springs in series: $keq = k_1k_2/(k_1 + k_2)$
Therefore: **$T = 2\pi\sqrt{\ [m\ (k_1 + k_2)\ /\ (k_1k_2)]}$**

4.10.2 Two Springs in Parallel

For two springs in parallel: $keq = k_1 + k_2$
Therefore: **$T = 2\pi\sqrt{[m/(k_1 + k_2)]}$**
These formulas apply to ideal springs and small oscillations about equilibrium.

4.11 Numerical Problems

Example 1: Time Period of a Simple Pendulum

A simple pendulum has a length of 1 m. Calculate its time period if $g = 9.8\ m\ s^{-2}$.

Solution

Given: $L = 1\ m$, $g = 9.8\ m\ s^{-2}$
Using: $T = 2\pi\sqrt{(L/g)} = 2\pi\sqrt{(1/9.8)}$
Therefore, **$T \approx 2.01\ s$**

Example 2: Effect of Increasing Length

The length of a simple pendulum is increased to four times its original value. Find the ratio of the new time period to the original time period.

Solution

We know: $T \propto \sqrt{L}$

Therefore: $T'/T = \sqrt{(L'/L)}$

Since $L' = 4L$,

$T'/T = \sqrt{4}$

Hence: $\mathbf{T'/T = 2}$

The new time period is twice the original time period.

Example 3: Length of a Seconds Pendulum

Calculate the length of a seconds pendulum for $g = 9.8\ m\ s^{-2}$.

Solution

Given: $T = 2\ s$, $g = 9.8\ m\ s^{-2}$

Using: $T = 2\pi\sqrt{(L/g)}$

$$2 = 2\pi\sqrt{(L/g)}$$

Therefore,

$$L = g/\pi^2 = 9.8/\pi^2$$

$L \approx 0.993\ m$

Example 4: Springs in Series

Two springs of spring constants $100\ N\ m^{-1}$ and $300\ N\ m^{-1}$ are connected in series. Calculate their effective spring constant.

Solution

Given: $k_1 = 100\ N\ m^{-1}$, $k_2 = 300\ N\ m^{-1}$

For series combination: $keq = k_1k_2/(k_1 + k_2)$

Substituting,

$keq = (100 \times 300)/(100 + 300) = 30000/400$ **$= 75\ N\ m^{-1}$**

Example 5: Springs in Parallel

Two springs of spring constants $100\ N\ m^{-1}$ and $300\ N\ m^{-1}$ are connected in parallel. Calculate their effective spring constant.

Solution

Given: $k_1 = 100\ N\ m^{-1}$, $k_2 = 300\ N\ m^{-1}$

For parallel combination: $keq = k_1 + k_2 = 100 + 300$ **$= 400\ N\ m^{-1}$**

Example 6: Time Period of a Mass Attached to Parallel Springs

A mass of 2 kg is attached to two springs of spring constants $100\ N\ m^{-1}$ and $300\ N\ m^{-1}$ connected in parallel. Calculate its time period.

Solution

Given: $m = 2\ kg$, $k_1 = 100\ N\ m^{-1}$, $k_2 = 300\ N\ m^{-1}$

For parallel springs: $keq = k_1 + k_2 = 400\ N\ m^{-1}$

Using: $T = 2\pi\sqrt{}\ (m/keq) = 2\pi\sqrt{(2/400)} = 2\pi\sqrt{0.005}$

Therefore: **$T \approx 0.444\ s$**