PART 1: Introduction to Wave Motion, Types of Waves and Basic Wave Parameters

1.1 Learning Objectives

After studying this part, students will be able to:

1. Understand the meaning and importance of wave motion.
2. Differentiate between the transfer of energy and the transfer of matter.
3. Define mechanical and electromagnetic waves.
4. Distinguish between transverse and longitudinal waves.
5. Understand the concepts of wavelength, frequency, time period and amplitude.
6. Establish the relationship between wave speed, frequency and wavelength.
7. Understand the basic characteristics of progressive waves.
8. Solve elementary numerical problems based on wave parameters.
9. Identify the applications of wave motion in daily life.

1.2 Introduction to Wave Motion

A wave is a disturbance that propagates through a medium or through space, transferring energy from one location to another.

In a mechanical wave, particles of the medium oscillate about their equilibrium positions as the disturbance travels through the medium.

Wave motion is a process of energy transfer without the overall transfer of matter along with the wave.

For example, when a stone is dropped into still water, circular ripples spread outward from the point of disturbance. The water particles mainly oscillate about their equilibrium positions, while the disturbance and energy travel outward.

Similarly, when one end of a stretched rope is moved up and down, a disturbance travels along the rope. The rope particles move up and down, but the disturbance travels along the rope.

Examples of Wave Motion

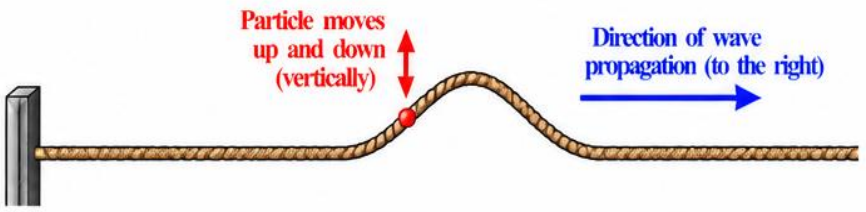
1. Waves travelling along a stretched string.
2. Sound waves travelling through air.
3. Water surface waves.
4. Seismic waves produced during earthquakes.
5. Electromagnetic waves, such as visible light and radio waves.

Important Characteristics of Wave Motion

- A wave transfers energy from one place to another.
- In mechanical waves, particles of the medium oscillate about their equilibrium positions.
- The wave travels through the medium, while the particles generally do not travel along with it over large distances.
- The nature of particle oscillation depends on the type of wave.
- Waves may undergo reflection, refraction, diffraction and interference.

Important: Mechanical waves require a material medium. Electromagnetic waves can propagate through a vacuum.

Figure 14.1: Wave Motion Along a Stretched Rope



A transverse wave pulse travelling to the right along a stretched rope.
Each particle of the rope moves up and down about its mean position.

1.3 Difference between Wave Motion and Particle Motion

Wave motion and particle motion are related, but they are not identical.
Consider a pulse travelling along a stretched rope.
The pulse moves from one end of the rope to the other, but each small segment of the rope moves mainly perpendicular to the direction of propagation.
Thus, the disturbance travels along the rope while the rope particles oscillate about their equilibrium positions.

Wave Motion	Particle Motion in a Mechanical Wave
The disturbance propagates through the medium.	Particles oscillate about their equilibrium positions in a progressive wave.
Energy is transported from one region to another.	Individual particles generally do not travel with the wave over large distances.
The propagation speed depends on the medium and wave properties.	The particle velocity varies with time and position.
A wave can travel a considerable distance.	In a simple progressive wave, each particle oscillates locally.

Example: In a sound wave travelling through air, air particles oscillate back and forth around their equilibrium positions while the sound disturbance travels through the air.

1.4 Classification of Waves

Waves can be classified according to the mechanism of propagation and the direction of particle oscillation.
The two important classifications are:

1. Mechanical and electromagnetic waves.
2. Transverse and longitudinal waves.

1.4.1 Mechanical Waves

Mechanical waves are waves that require a material medium for propagation.
They arise due to disturbances in a medium and are transmitted through interactions between its particles.

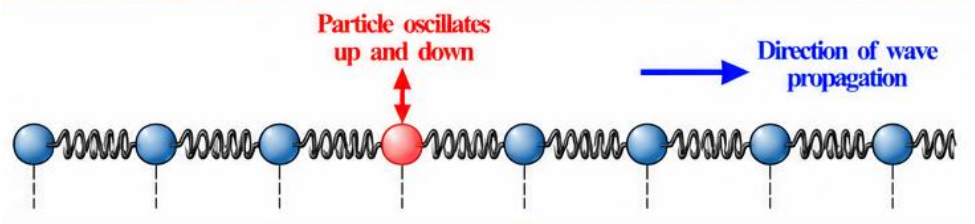
Examples

- Sound waves in air.
- Waves on a stretched string.
- Seismic waves.
- Waves travelling through a solid rod.

Important Properties

1. Mechanical waves cannot propagate through a perfect vacuum.
2. Their speed depends on the properties of the medium.
3. The medium's particles oscillate about their equilibrium positions.
4. Mechanical waves may be transverse, longitudinal or more complicated, depending on the medium and conditions.

Figure 14.2: Mechanical Wave Travelling Through a Medium



A mechanical transverse wave travelling through a medium.
The particles of the medium oscillate about their mean positions.

1.4.2 Electromagnetic Waves

Electromagnetic waves consist of oscillating electric and magnetic fields that propagate through space.

Unlike mechanical waves, electromagnetic waves do not require a material medium.

Examples

- Radio waves.
- Microwaves.
- Infrared radiation.
- Visible light.
- Ultraviolet radiation.
- X-rays.
- Gamma rays.

In vacuum, electromagnetic waves travel at the speed of light: $\mathbf{c \approx 3 \times 10^8\ m\ s^{-1}}$

In a vacuum, their frequency and wavelength are related by: $\mathbf{c = f\lambda}$

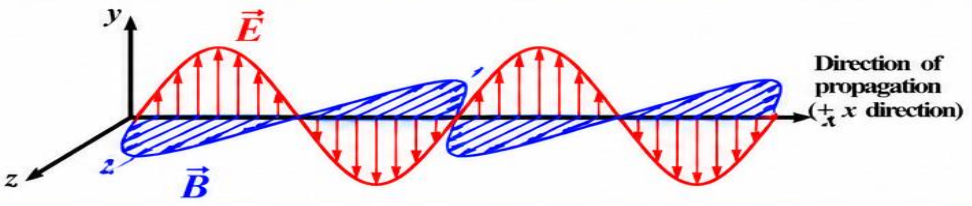
Where:

- c = speed of light in vacuum
- f = frequency
- λ = wavelength.

Electromagnetic waves are transverse in nature in the usual description of plane waves in free space.

The electric field, magnetic field and direction of propagation are mutually perpendicular in a plane electromagnetic wave in vacuum.

Figure 14.3: Electromagnetic Wave



An electromagnetic wave propagating along the $+x$ direction. The electric field $\vec{E}$ oscillates along the $+y$ direction and the magnetic field $\vec{B}$ oscillates along the $+z$ direction. $\vec{E}$, $\vec{B}$ and the direction of propagation are mutually perpendicular.

1.5 Transverse Waves

A transverse wave is a wave in which particles of the medium oscillate perpendicular to the direction of wave propagation.

For example, consider a wave travelling along a stretched horizontal string.

If the wave travels horizontally while the string particles move vertically, the wave is transverse.

Examples

1. Waves on a stretched string.
2. Electromagnetic waves in vacuum.
3. Certain seismic waves, such as S-waves, travelling through solids.

Important Terms

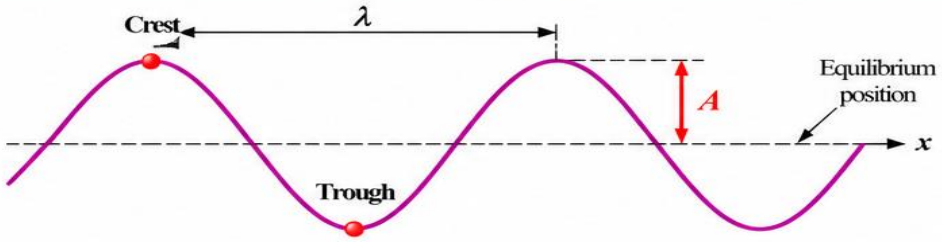
Crest: The highest point of a transverse wave above its equilibrium position.

Trough: The lowest point of a transverse wave below its equilibrium position.

Amplitude: The maximum displacement of a particle from its equilibrium position.

Wavelength: The distance between two consecutive crests or two consecutive troughs.

Figure 14.4: Transverse Wave Showing Crest, Trough and Amplitude



A transverse wave showing a crest, a trough and the amplitude A. The wavelength λ is the distance between two successive crests (or troughs).

1.6 Longitudinal Waves

A longitudinal wave is a wave in which particles of the medium oscillate parallel to the direction of wave propagation.

When a longitudinal wave travels through air, the air particles move back and forth along the direction in which the sound travels.

This creates alternating regions of compression and rarefaction.

Compression

A compression is a region in which particles of the medium are closer together than their average separation.

Pressure and density are generally higher in a compression than in the undisturbed medium.

Rarefaction

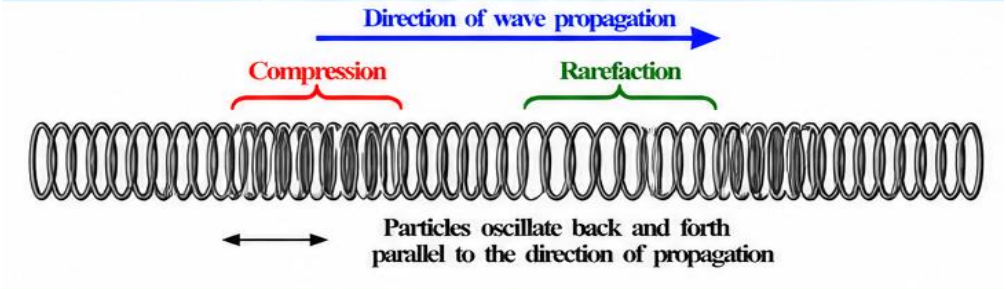
A rarefaction is a region in which particles of the medium are farther apart than their average separation.

Pressure and density are generally lower in a rarefaction than in the undisturbed medium.

Examples

1. Sound waves in air.
2. Compression waves travelling through a spring.
3. P-waves generated during earthquakes.

Figure 14.5: Longitudinal Wave Showing Compressions and Rarefactions



A longitudinal wave showing compressions (closely spaced coils) and rarefactions (widely spaced coils). The particles of the medium oscillate parallel to the direction of propagation.

1.7 Difference Between Transverse and Longitudinal Waves

Transverse Waves	Longitudinal Waves
Particle oscillations are perpendicular to the direction of propagation.	Particle oscillations are parallel to the direction of propagation.
Crests and troughs may be represented in a displacement diagram.	Compressions and rarefactions are characteristic features.
Waves on a stretched string are an example.	Sound waves in air are an example.
Transverse mechanical waves require suitable restoring forces in the medium.	Longitudinal mechanical waves can propagate through solids, liquids and gases.
Electromagnetic waves in vacuum are transverse.	Sound waves in air are longitudinal.

Important: A transverse displacement graph is a representation of displacement against position. For a longitudinal wave, such a graph can also be drawn, but it represents the longitudinal displacement of particles rather than the physical shape of the medium.

1.8 Important Wave Parameters

To describe a periodic wave mathematically, we need several physical quantities.

These include:

1. Amplitude.
2. Wavelength.
3. Time period.
4. Frequency.
5. Wave speed.
6. Phase.

1.8.1 Amplitude (A)

Amplitude is the maximum displacement of a particle of the medium from its equilibrium position.
It is represented by A.
The SI unit of amplitude is the metre (m).
For a transverse wave, amplitude is measured perpendicular to the direction of propagation.
For a longitudinal wave, particle displacement is measured parallel to the direction of propagation.
Important: Amplitude is not the same as wavelength.

1.8.2 Wavelength (λ)

Wavelength is the distance between two successive points that are in the same phase of oscillation.
For a transverse wave, wavelength can be measured as the distance between:

- Two successive crests.
- Two successive troughs.

For a longitudinal wave, wavelength is the distance between:

- Two successive compressions.
- Two successive rarefactions.

The SI unit of wavelength is the metre (m).
Symbol: λ (lambda)

1.8.3 Time Period (T)

The time period of a wave is the time taken by a particle of the medium to complete one full oscillation.
The SI unit of time period is the second (s).
For a periodic wave, the time period is also the time taken by the wave pattern to advance by one wavelength.

1.8.4 Frequency (f)

Frequency is the number of complete oscillations performed by a particle per unit time.
If N oscillations occur in time t, then, **f = N/t**
The SI unit of frequency is the hertz (Hz).
One hertz corresponds to one complete oscillation per second.
1 Hz = 1 s^{-1}
Frequency and time period are related by: **f = 1/T** or **T = 1/f**

1.8.5 Wave Speed (v)

Wave speed is the distance travelled by a wave disturbance per unit time.
If a wave travels a distance d in time t, its average propagation speed is v = d/t
For a periodic wave, the disturbance travels one wavelength λ in one time period T.
Therefore, v = λ/T
Since f = 1/T, **v = fλ**
This is the fundamental relationship between wave speed, frequency and wavelength.

Important Formula

Wave speed = Frequency × Wavelength

$$\mathbf{v = f\lambda}$$

Where:

- v = wave speed, in m s^{-1}
- f = frequency, in Hz
- λ = wavelength, in m.

Dimensional Verification

The dimension of frequency is $[f] = T^{-1}$
The dimension of wavelength is $[\lambda] = L$
Therefore, $[f\lambda] = LT^{-1}$
This is the dimension of speed.
Hence, the equation $v = f\lambda$ is dimensionally correct.

1.9 Derivation of the Wave-Speed Relation

Consider a periodic wave travelling through a medium.
Let:

- λ = wavelength
- T = time period
- f = frequency
- v = wave speed.

During one complete time period T, the wave pattern advances by one wavelength λ.
Therefore,
Distance travelled = λ
Time taken = T
Using speed = distance/time,

$$v = \lambda/T$$

But

$$f = 1/T$$

Substituting,

$$\mathbf{v = f\lambda}$$

Thus, the wave-speed relation is established.

Important Physical Meaning

For a given wave speed in a particular medium, $v = f\lambda$
Therefore, $\boldsymbol{\lambda = v/f}$ and $\boldsymbol{f = v/\lambda}$
If the wave speed remains constant, increasing the frequency decreases the wavelength.
Similarly, decreasing the frequency increases the wavelength.

1.10 Basic Characteristics of Progressive Waves

A progressive wave is a disturbance that travels continuously through a medium, transferring energy from one location to another.

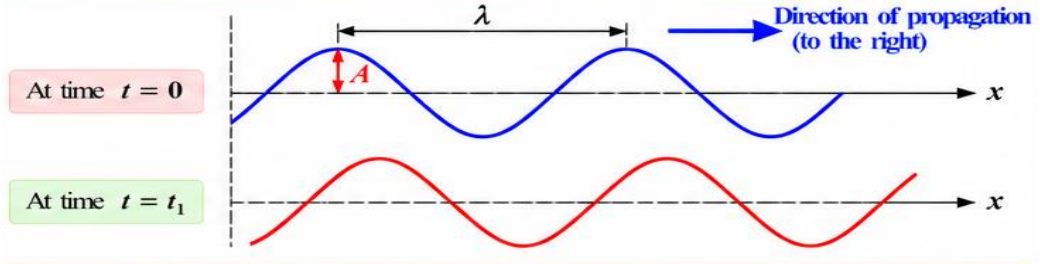
Important Characteristics

1. The wave disturbance travels through the medium.
2. Particles of the medium oscillate about their equilibrium positions.
3. In an ideal progressive harmonic wave, all particles have the same frequency.
4. In a uniform, non-dispersive medium, the wave speed is independent of the frequency.
5. Different particles generally have different phases at a given instant.

6. The energy is transported by the wave.
7. In an ideal undamped wave, the amplitude does not decrease because of energy dissipation.

In a real medium, the amplitude may decrease because of absorption, scattering or geometric spreading.

Figure 14.6: Progressive Wave Travelling Along a String



A progressive wave travelling to the right along a string.
The lower curve shows the wave profile at a later time t_1. The wave moves to the right without change in shape. Here A is the amplitude and λ is the wavelength.

1.11 Phase Difference Between Two Points on a Wave

Phase describes the stage of oscillation of a particle at a particular instant.

Two particles are in the same phase if their displacements and directions of motion are the same at the corresponding instant.

For a sinusoidal progressive wave, the phase difference between two points separated by distance Δx is **$\Delta\varphi = 2\pi\Delta x/\lambda$**

Where:

- Δφ = phase difference in radians
- Δx = separation between the points along the direction of propagation
- λ = wavelength.

This relation assumes that the points belong to the same sinusoidal wave at the same instant.

Important Cases

Case 1: Separation = λ; $\Delta\varphi = 2\pi$

The points are in the same phase.

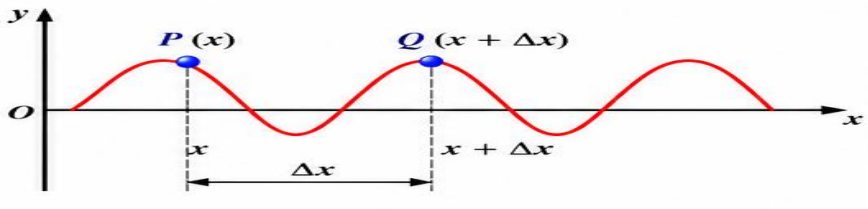
Case 2: Separation = λ/2; $\Delta\varphi = \pi$

The points differ in phase by 180°.

Case 3: Separation = λ/4; $\Delta\varphi = \pi/2$

The points differ in phase by 90°.

Figure 14.7: Phase Difference Between Two Points on a Wave



Two points P and Q on a wave separated by a distance Δx have a phase difference $\Delta\varphi = \frac{2\pi}{\lambda}\Delta x = k\Delta x$.

Example

Two points on a sinusoidal wave are separated by 0.5 m. The wavelength is 2 m. Find their phase difference.

Solution:

Given: $\Delta x = 0.5$ m, $\lambda = 2$ m

Using,

$$\Delta\varphi = 2\pi\Delta x/\lambda = 2\pi \times 0.5/2$$

Therefore, **Phase difference = $\pi/2$ rad = 90°**

1.12 Progressive Wave Equation: Introduction

A progressive sinusoidal wave can be represented mathematically by a displacement equation. For a wave travelling in the positive x-direction, one common form is

$$\mathbf{y(x, t) = A \sin(kx - \omega t + \varphi)}$$

Where:

- y(x,t) = displacement of the particle at position x and time t
- A = amplitude
- k = wave number
- ω = angular frequency
- φ = initial phase constant.

The wave number is defined as $\mathbf{k = 2\pi/\lambda}$

The angular frequency is $\boldsymbol{\omega = 2\pi f}$

The wave speed is $\mathbf{v = \omega/k}$

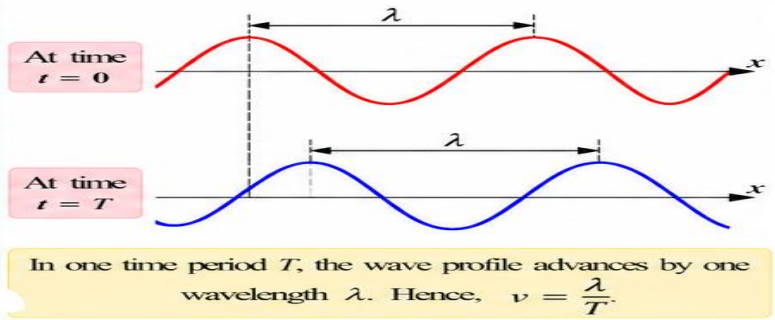
Substituting the definitions of ω and k, we get

$$v = (2\pi f)/(2\pi/\lambda)$$

Therefore,

$$\mathbf{v = f\lambda}$$

Figure 14.8: Wavelength and Time Period of a Progressive Wave



Direction of Propagation

For the form

$$y(x, t) = A \sin(kx - \omega t + \varphi),$$

The wave travels in the positive x-direction.

For the form

$$y(x, t) = A \sin(kx + \omega t + \varphi),$$

The wave travels in the negative x-direction.

Important: The signs depend on the chosen phase convention. The expressions above use the standard sine-wave convention.

The detailed derivation and applications of the progressive wave equation will be covered in the next part.

1.13 Solved Numerical Problems

Example 1: Finding Wave Speed

A wave has a frequency of 50 Hz and a wavelength of 4 m. Calculate its speed.

Solution

Given: $f = 50$ Hz, $\lambda = 4$ m

Using: $v = f\lambda = 50 \times 4$

Therefore, $\mathbf{v = 200\ m\ s^{-1}}$

Example 2: Finding Wavelength

A wave travels with a speed of 300 m s^{-1} and has a frequency of 150 Hz. Calculate its wavelength.

Solution

Given: $v = 300$ m s^{-1}, $f = 150$ Hz

Using: $\lambda = v/f = 300/150$

Therefore, $\boldsymbol{\lambda = 2\ m}$

Example 3: Finding Frequency

A wave travels through a medium with a speed of 340 m s^{-1} and has a wavelength of 0.68 m. Calculate its frequency.

Solution

Given: $v = 340$ m s^{-1}, $\lambda = 0.68$ m

Using: $f = v/\lambda = 340/0.68$

Therefore, $\mathbf{f = 500\ Hz}$

Example 4: Finding Time Period

A sound wave has a frequency of 250 Hz. Calculate its time period.

Solution

Given: $f = 250$ Hz

Using,

$$T = 1/f = 1/250$$

Therefore, $\mathbf{T = 0.004\ s \text{ or } T = 4\ ms}$

Example 5: Finding Phase Difference

Two points on a progressive sinusoidal wave are separated by 0.75 m. The wavelength is 3 m. Calculate their phase difference.

Solution

Given: $\Delta x = 0.75$ m, $\lambda = 3$ m

Using,

$$\Delta\varphi = 2\pi\Delta x/\lambda = 2\pi \times 0.75/3 = \pi/2 \text{ rad}$$

Therefore, $\mathbf{\text{Phase difference} = 90°}$

PART 2: Progressive Wave Equation, Particle Velocity and Acceleration, and Speed of Transverse Waves

2.1 Learning Objectives

After studying this part, students will be able to:

1. Understand the propagation of a progressive wave.
2. Derive the mathematical equation of a progressive sinusoidal wave.
3. Identify the direction of propagation from the wave equation.
4. Explain the relationship between wave number, wavelength and angular frequency.
5. Derive expressions for particle velocity and acceleration in a progressive wave.
6. Distinguish between particle velocity and wave propagation speed.
7. Derive the speed of transverse waves on a stretched string.
8. Understand the factors affecting wave speed on a string.
9. Solve numerical problems involving progressive waves and stretched strings.

2.2 Progressive Waves

A progressive wave is a disturbance that travels continuously through a medium, transferring energy from one point to another.

In a progressive mechanical wave, particles of the medium oscillate about their equilibrium positions as the disturbance passes through them.

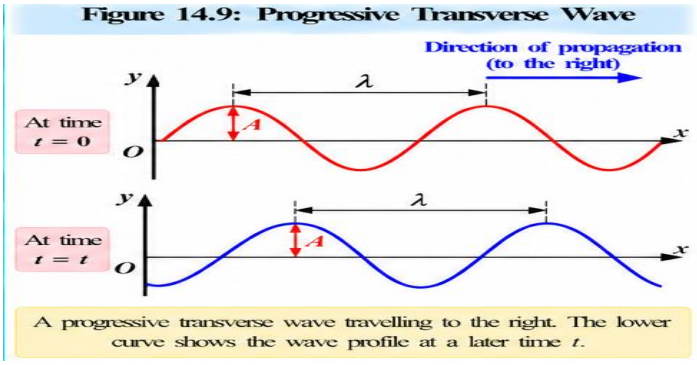
A progressive wave may travel in either direction along a given axis.

Important Characteristics

1. The disturbance propagates through the medium.
2. The particles of the medium oscillate about their equilibrium positions.
3. In an ideal progressive sinusoidal wave, all particles have the same frequency.
4. Particles at different positions generally have different phases.
5. Energy is transferred through the medium.
6. In an ideal, uniform, non-dispersive medium without attenuation, the wave speed is constant.
7. The wave speed depends on the properties of the medium.

Examples

- A sinusoidal wave travelling along a stretched string.
- Sound waves travelling through air.
- A disturbance travelling through a spring.
- Seismic waves travelling through the Earth.



2.3 Mathematical Representation of a Progressive Wave

Consider a sinusoidal wave travelling along the positive x-direction.
Let:

- A = amplitude
- λ = wavelength
- T = time period
- f = frequency
- ω = angular frequency
- k = wave number
- v = wave propagation speed.

Suppose the displacement of a particle at the origin is **$y(0,t) = A \sin \omega t$**
A particle at position x oscillates with the same amplitude and frequency, but its phase differs from that of the particle at the origin.
For a wave travelling in the positive x-direction, the displacement equation can be written as

$$\mathbf{y(x,t) = A \sin(\omega t - kx)}$$

An equivalent form is

$$\mathbf{y(x,t) = A \sin(kx - \omega t + \pi)}$$

depending on the chosen phase convention.
A particularly common representation is

$$\mathbf{y(x,t) = A \sin(kx - \omega t + \varphi)}$$

where φ is the initial phase constant.
For a wave travelling in the negative x-direction, one common form is

$$\mathbf{y(x,t) = A \sin(kx + \omega t + \varphi)}$$

The sign of the time-dependent term determines the direction of propagation when the remaining conventions are held fixed.
Important: The positive- and negative-direction equations may look different depending on whether sine or cosine is used. Always identify the direction by examining the phase.

2.4 Derivation of the Progressive Wave Equation

Consider a sinusoidal transverse wave travelling along a stretched string.
Let the displacement of a particle at the origin be **$y(0,t) = A \sin \omega t$**
Now consider a particle at position x.
The wave takes time **$\Delta t = x/v$**
to travel from the origin to that position, where v is the wave speed.
Therefore, the displacement of the particle at position x and time t is equal to the displacement that the particle at the origin had at the earlier time $t - x/v$.
Thus,

$$y(x, t) = A \sin[\omega(t - x/v)]$$

Expanding,

$$y(x, t) = A \sin(\omega t - \omega x/v)$$

We know that

$$v = f\lambda$$

And

$$\omega = 2\pi f.$$

The wave number is

$$k = 2\pi/\lambda.$$

Therefore,

$$\omega/v = (2\pi f)/(f\lambda)$$

Hence,

$$\omega/v = 2\pi/\lambda = k.$$

Substituting,

$$y(x,t) = A \sin(\omega t - kx)$$

Therefore, the progressive wave equation for this choice of initial phase is

$$\mathbf{y(x,t) = A \sin(\omega t - kx)}$$

This equation represents a wave travelling in the positive x-direction.
An equivalent representation is

$$\mathbf{y(x,t) = A \sin(kx - \omega t + \pi)}$$

since $\sin(\omega t - kx) = -\sin(kx - \omega t)$.

General Form

The general sinusoidal wave equation can be written as

$$\mathbf{y(x, t) = A \sin(kx - \omega t + \varphi)}$$

for propagation in the positive x-direction, or

$$\mathbf{y(x, t) = A \sin(kx + \omega t + \varphi)}$$

for propagation in the negative x-direction.
Here, φ is the phase constant.

2.5 Wave Number (k)

Wave number is a quantity that describes the spatial rate of phase change of a wave.
It is defined as $\mathbf{k = 2\pi/\lambda}$
where λ is the wavelength.

SI Unit

The SI unit of wave number is m^{-1}. It is also commonly written as rad m^{-1} when the phase interpretation is emphasized.

Physical Meaning

A smaller wavelength corresponds to a larger wave number.
A larger wavelength corresponds to a smaller wave number.
For example, if the wavelength is halved, the wave number doubles.
Important: In wave motion, k represents the wave number. In spring problems, the symbol k often represents the spring constant. Their meanings depend on context.

2.6 Angular Frequency (ω)

Angular frequency describes the rate at which the phase of an oscillation changes with time.
It is defined as $\mathbf{\omega = 2\pi f}$
Since $f = 1/T$,
We also have $\mathbf{\omega = 2\pi/T}$
The SI unit is rad s^{-1}.

Relationship Between Wave Speed, Angular Frequency and Wave Number

We know that $v = f\lambda$.
Using

$$\omega = 2\pi f$$

And

$$k = 2\pi/\lambda,$$

We obtain

$$\omega/k = (2\pi f)/(2\pi/\lambda)$$

Therefore,

$$\mathbf{v = \omega/k}$$

This is another important expression for wave propagation speed.

Summary of Wave Parameters

Quantity	Formula	SI Unit
Frequency	$f = 1/T$	Hz
Angular frequency	$\omega = 2\pi f$	rad s^{-1}
Wavelength	$\lambda = v/f$	m
Wave number	$k = 2\pi/\lambda$	m^{-1}
Wave speed	$v = f\lambda$	m s^{-1}
Wave speed	$v = \omega/k$	m s^{-1}

2.7 Direction of Propagation of a Wave

The direction of propagation can be identified from the phase of the wave equation.

Consider

$$\mathbf{y(x, t) = A \sin(kx - \omega t)}$$

For a point of constant phase,

$$kx - \omega t = \text{constant}.$$

Differentiating with respect to time,

$$k(dx/dt) - \omega = 0.$$

Therefore,

$$dx/dt = \omega/k.$$

Since ω/k is positive, the constant-phase point moves in the positive x-direction.

Hence, the wave travels in the positive x-direction.

Now consider

$$\mathbf{y(x, t) = A \sin(kx + \omega t)}$$

For a point of constant phase,

$$kx + \omega t = \text{constant}.$$

Differentiating,

$$k(dx/dt) + \omega = 0.$$

Therefore,

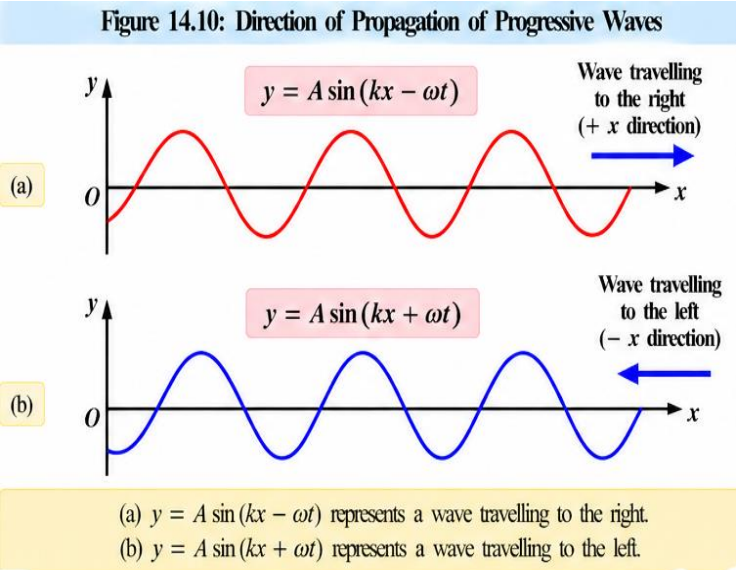
$$dx/dt = -\omega/k.$$

The negative sign indicates motion in the negative x-direction.

Important Results

- $y = A \sin(kx - \omega t)$ represents propagation in the positive x-direction.
- $y = A \sin(kx + \omega t)$ represents propagation in the negative x-direction.

These results apply when A, k and ω are positive and the phase is written in the stated convention.



2.8 Particle Velocity in a Progressive Wave

Particle velocity is the velocity of an individual particle of the medium as it oscillates.
It is different from the wave propagation speed.
Consider the progressive wave equation

$$y(x,t) = A \sin(kx - \omega t).$$

Particle velocity is obtained by differentiating displacement with respect to time while keeping x constant.
Therefore, $u = \partial y/\partial t$
Differentiating,

$$\mathbf{u = -A\omega \cos(kx - \omega t)}$$

This is the particle velocity for the stated wave equation.

Maximum Particle Speed

The maximum magnitude of cos (kx − ωt) is 1.
Therefore,

$$\boxed{u_{max} = A\omega}$$

Since ω = 2πf,

$$u_{max} = 2\pi Af$$

Important Observations

1. Particle velocity varies periodically with time.
2. Its maximum magnitude is Aω.
3. Particle velocity may be positive or negative.
4. The particle velocity is not generally equal to the wave propagation speed.
5. For a transverse wave on a string, particle velocity is perpendicular to the direction of wave propagation.

Important: The symbol u is used here for particle velocity to distinguish it from wave speed v.

2.9 Particle Acceleration in a Progressive Wave

Particle acceleration is the rate of change of particle velocity with time.
For the wave: $y(x, t) = A \sin (kx - \omega t)$,

The particle velocity is: $u = -A\omega \cos(kx - \omega t)$.
Differentiating with respect to time while keeping x constant,

$$a = \partial u/\partial t.$$

Therefore,

$$\mathbf{a = -A\omega^2 \sin(kx - \omega t)}$$

Since: $y = A \sin(kx - \omega t)$,
We obtain:

$$\boxed{a = -\omega^2 y}$$

This means that, at any fixed position, the particle's transverse acceleration is proportional to its displacement from equilibrium and opposite in direction.
Thus, each particle of an ideal sinusoidal transverse wave executes SHM as the wave passes, provided the wave motion is small enough for the linear model to apply.

Maximum Particle Acceleration

The maximum magnitude of the sine function is 1.
Therefore: $\boxed{a_{max} = A\omega^2}$

Important Observations

- Particle acceleration is zero when the particle displacement is zero.
- The magnitude of acceleration is maximum at the extreme displacements.
- Acceleration is directed opposite to displacement.
- The particle motion is SHM for an ideal sinusoidal progressive wave.

2.10 Particle Velocity and Wave Speed: Difference

This distinction is very important for CBSE, NEET and JEE examinations.

Particle Velocity	Wave Propagation Speed
It is the velocity of a particle of the medium.	It is the speed at which the disturbance travels.
It varies with time and position.	It is constant for a given wave in an ideal uniform, non-dispersive medium.
For a transverse wave, particle motion is perpendicular to propagation.	It is directed along the direction of propagation.
Its maximum magnitude is $A\omega$ for a sinusoidal wave.	Its magnitude is $v = f\lambda = \omega/k$.
It describes the motion of the medium's particles.	It describes the motion of the wave pattern.

Example

A wave travels along a string at 20 m s^{-1}.
This does not mean that every string particle moves at 20 m s^{-1}. Each particle oscillates locally, and its velocity changes continuously.

2.11 Speed of Transverse Waves on a Stretched String

The speed of a transverse wave on an ideal stretched string depends on the tension in the string and its mass per unit length.
The wave speed is $\mathbf{v = \sqrt{(T/\mu)}}$
Where:

- v = transverse wave speed

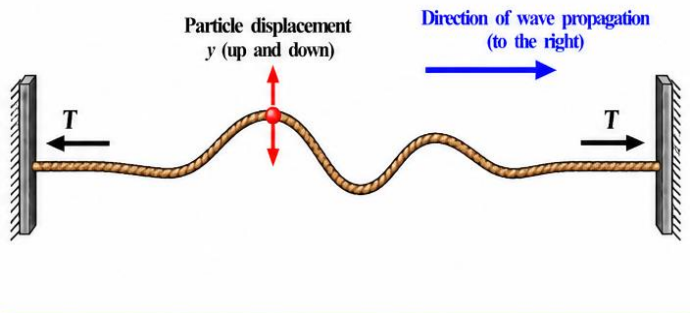
- T = tension in the string
- μ = mass per unit length.

The mass per unit length is **μ = m/L**
Where, m is the mass of the string and L is its length.
The SI unit of μ is kg m^{-1}.

Important Conclusions

1. Wave speed increases when the tension increases.
2. Wave speed decreases when the mass per unit length increases.
3. Wave speed is independent of amplitude in the ideal linear model.
4. Wave speed is determined by the properties of the medium rather than by the frequency alone.

Figure 14.11: Transverse Wave on a Stretched String



A transverse wave travelling along a stretched string fixed at both ends. *T* is the tension in the string, the wave propagates to the right, and each particle of the string moves up and down about its mean position.

2.12 Derivation of the Speed of a Transverse Wave on a String

Consider a flexible string stretched under tension F_T.
Let the mass per unit length of the string be μ.
A small transverse disturbance travels along the string.
For a small segment of the string, the net transverse force arises from the difference between the directions of tension at its two ends.
For small slopes, the vertical components of tension can be approximated using the local slope of the string.
The resulting transverse force on a small segment is approximately proportional to the second spatial derivative of displacement.
Applying Newton's second law gives the ideal string-wave equation:

$$\partial^2 y/\partial t^2 = (T/\mu)(\partial^2 y/\partial x^2)$$

For a sinusoidal progressive wave,

$$y = A \sin(kx - \omega t),$$

We have

$$\partial^2 y/\partial t^2 = -\omega^2 y$$

And

$$\partial^2 y/\partial x^2 = -k^2 y.$$

Substituting into the wave equation,

$$-\omega^2 y = (T/\mu)(-k^2 y).$$

Cancelling the common factor $-y$,

$$\omega^2 = (T/\mu)k^2.$$

Therefore,

$$\omega^2/k^2 = T/\mu.$$

Since $v = \omega/k$,

$$v^2 = T/\mu.$$

Taking the positive square root for the magnitude of wave speed,

$$\mathbf{v = \sqrt{(T/\mu)}}$$

This is the required expression for the speed of a transverse wave on an ideal stretched string.

Assumptions

The derivation assumes:

1. The string is flexible and uniform.
2. The tension is approximately constant.
3. The transverse displacement and slope are small.
4. The string's mass per unit length is uniform.
5. Damping and other losses are neglected.

2.13 Effect of Tension and Linear Mass Density

The wave speed on a stretched string is:

$$v = \sqrt{(T/\mu)}.$$

2.13.1 Effect of Tension

For constant μ,

$$v \propto \sqrt{T}.$$

If the tension becomes four times its original value,

$$v' = \sqrt{(4T/\mu)}.$$

Therefore,

$$\mathbf{v' = 2v}$$

Thus, quadrupling the tension doubles the wave speed.

2.13.2 Effect of Linear Mass Density

For constant tension,

$$v \propto 1/\sqrt{\mu}.$$

If the linear mass density becomes four times its original value,

$$v' = \sqrt{[T/(4\mu)]}.$$

Therefore,

$$\mathbf{v' = v/2}$$

Thus, quadrupling the mass per unit length halves the wave speed.

General Ratio Formula

For two strings under different tensions and with different linear mass densities,

$$\mathbf{v_2/v_1 = \sqrt{[(T_2\ \mu_1)\ /\ (T_1\ \mu_2)]}}$$

This formula is useful for comparative numerical problems.

2.14 Solved Numerical Problems

Example 1: Wave Speed from Frequency and Wavelength

A progressive wave has a frequency of 200 Hz and a wavelength of 1.5 m. Calculate its speed.

Solution

Given: f = 200 Hz, λ = 1.5 m

Using,

$$v = f\lambda = 200 \times 1.5 = \mathbf{300\ m\ s^{-1}}$$

Example 2: Particle Velocity

A transverse wave is represented by y = 0.04 sin(5x − 20t), where x and y are in metres and t is in seconds. Calculate the maximum particle speed.

Solution

Comparing with y = A sin(kx − ωt),

We obtain: A = 0.04 m, ω = 20 rad s^{-1}

The maximum particle speed is

$$u_{max} = A\omega = 0.04 \times 20 = \mathbf{0.8\ m\ s^{-1}}$$

Example 3: Wave Speed from the Wave Equation

A progressive wave is represented by y = 0.02 sin (4x − 12t). Calculate its wavelength, frequency and propagation speed.

Solution

Comparing with: y = A sin (kx − ωt),

We obtain:

k = 4 rad m^{-1}, ω = 12 rad s^{-1}.

Step 1: Wavelength

$$k = 2\pi/\lambda$$

Therefore:

$\lambda = 2\pi/k = 2\pi/4 = \mathbf{\pi/2\ m \approx 1.57\ m}$

Step 2: Frequency

$$\omega = 2\pi f$$

Therefore:

$f = \omega/(2\pi) = 12/(2\pi) \approx \mathbf{1.91\ Hz}$

Step 3: Wave Speed

$$v = \omega/k = 12/4 = \mathbf{3\ m\ s^{-1}}$$

Example 4: Wave Speed on a Stretched String

A string has a tension of 100 N and a linear mass density of 0.01 kg m^{-1}. Calculate the transverse wave speed.

Solution

Given: T = 100 N, μ = 0.01 kg m^{-1}

Using,

$$v = \sqrt{(T/\mu)} = \sqrt{(100/0.01)} = \sqrt{10000} = \mathbf{100\ m\ s^{-1}}$$

Example 5: Effect of Tension

A transverse wave travels along a string at 20 m s^{-1}. If the tension is increased to four times its original value while the linear mass density remains constant, calculate the new wave speed.

Solution

For a stretched string:

$$\boxed{v \propto \sqrt{T}}$$

Therefore,

$$v'/v = \sqrt{(T'/T)}$$

Since $T' = 4T$,
Hence,

$$v'/v = \sqrt{4} = 2.$$

Thus,

$$v' = 2 \times 20 = \mathbf{40\ m\ s^{-1}}$$

Example 6: Linear Mass Density

A string of length 2 m has a mass of 0.04 kg. It is stretched under a tension of 100 N. Calculate the wave speed.

Solution

Given: m = 0.04 kg, L = 2 m, T = 100 N
Linear mass density:

$$\mu = m/L = 0.04/2 = 0.02\ kg\ m^{-1}$$

Wave speed:

$$v = \sqrt{(T/\mu)} = \sqrt{(100/0.02)} = \sqrt{5000}$$

Therefore, $\mathbf{v \approx 70.7\ m\ s^{-1}}$

PART 3: SUPERPOSITION OF WAVES, INTERFERENCE, REFLECTION OF WAVES AND STANDING WAVES

3.1 Principle of Superposition of Waves

Definition

The **principle of superposition of waves** states that when two or more waves travel through the same medium simultaneously, the resultant displacement of any particle of the medium is equal to the algebraic sum of the displacements produced by the individual waves.
For two waves producing displacements y_1 and y_2 at a particular point and time:
Resultant displacement:

$$y = y_1 + y_2$$

For three waves:

$$y = y_1 + y_2 + y_3$$

In general:

$$\mathbf{y = \Sigma y_i}$$

This principle applies to waves in a linear medium, provided the disturbances are small enough for the medium to behave linearly.

Important Points

1. The individual waves continue to propagate independently through the medium.
2. The resultant displacement depends on the phase difference between the waves.
3. The resultant amplitude may be greater than, less than, or equal to the amplitude of an individual wave.
4. After overlapping, the waves continue their propagation without permanently changing their individual identities in a linear medium.

Example

Suppose two waves produce displacements at a point: $y_1 = +3$ cm, $y_2 = -2$ cm
Then the resultant displacement is:

$$y = y_1 + y_2 = 3 - 2 = \mathbf{+1\ cm}$$

The particle is displaced by 1 cm in the positive direction from its equilibrium position.

3.2 Superposition of Two Waves Having the Same Frequency

Consider two waves of equal frequency travelling through the same medium:
$y_1 = A_1 \sin \omega t$ and $y_2 = A_2 \sin (\omega t + \varphi)$
Where:

- A_1 and A_2 are the amplitudes of the two waves.
- ω is their common angular frequency.
- φ is the phase difference between them.

According to the principle of superposition:

$$y = y_1 + y_2$$

Therefore,

$$y = A_1 \sin \omega t + A_2 \sin(\omega t + \varphi)$$

The resultant wave can be expressed as:

$$\mathbf{y = A \sin (\omega t + \delta)}$$

Where, A is the resultant amplitude and δ is the phase of the resultant wave.
The resultant amplitude is given by: $\mathbf{A = \sqrt{(A_1^2 + A_2^2 + 2A_1A_2 \cos \varphi)}}$

This formula is important for understanding constructive and destructive interference.

Special Cases

Case 1: Waves are in the same phase

If $\varphi = 0$, then $\cos \varphi = 1$.

$$A = \sqrt{(A_1^2 + A_2^2 + 2A_1A_2)}$$

Or,

$$\mathbf{A = A_1 + A_2}$$

The resultant amplitude is maximum.

Case 2: Waves are in opposite phases

If $\varphi = \pi$, then $\cos \varphi = -1$.

$$A = \sqrt{(A_1^2 + A_2^2 - 2A_1A_2)}$$

Or,

$$\mathbf{A = |A_1 - A_2|}$$

The resultant amplitude is minimum.

If $A_1 = A_2$, the resultant displacement is zero at every point where the two waves have exactly opposite phase.

Case 3: Phase difference is $\pi/2$

If $\varphi = \pi/2$, then $\cos \varphi = 0$.

$$\mathbf{A = \sqrt{(A_1^2 + A_2^2)}}$$

For equal amplitudes $A_1 = A_2 = a$:

$$A = \sqrt{2a^2}$$

Or,

$$\boldsymbol{A = a\sqrt{2}}$$

3.3 Interference of Waves

Definition

Interference is the phenomenon in which two or more coherent waves overlap and produce a resultant intensity distribution containing regions of reinforcement and cancellation.

Interference is a direct consequence of the principle of superposition.

For sustained, clearly defined interference, the waves should generally have:

1. The same frequency.
2. A constant phase difference.
3. The same state of polarization, in the case of transverse electromagnetic waves, for maximum contrast.

For mechanical waves such as sound waves, the sources must maintain a stable phase relationship.

Constructive Interference

Constructive interference occurs when the two waves arrive in the same phase.

For constructive interference:

$$\textbf{Path difference, } \Delta x = n\lambda$$

Or,

$$\textbf{Phase difference, } \varphi = 2n\pi$$

Where, n = 0, 1, 2, 3, ...

For two waves of equal amplitude a: A = a + a **= 2a**

Thus, the resultant amplitude is maximum.

Since intensity is proportional to the square of amplitude:

$$I \propto A^2$$

For equal individual intensities I_0:

$$I_{max} = 4I_0$$

Destructive Interference

Destructive interference occurs when the two waves arrive in opposite phases.
The condition is:

$$\textbf{Path difference, } \Delta x = (2n + 1)\lambda/2$$

or

$$\textbf{Phase difference, } \varphi = (2n + 1)\pi$$

For two waves of equal amplitude a:

$$A = |a - a| = 0$$

Therefore, complete cancellation occurs at that point in the ideal case.
For equal individual intensities:

$$I_{min} = 0$$

If the amplitudes are unequal, destructive interference does not generally produce zero resultant intensity.

Interference Formula for Intensity

For two coherent waves having individual intensities I_1 and I_2 and phase difference φ:

$$I = I_1 + I_2 + 2\sqrt{(I_1 I_2)} \cos \varphi$$

This formula is useful in problems involving interference.
For constructive interference, cos φ = 1:

$$I_{max} = (\sqrt{I_1} + \sqrt{I_2})^2$$

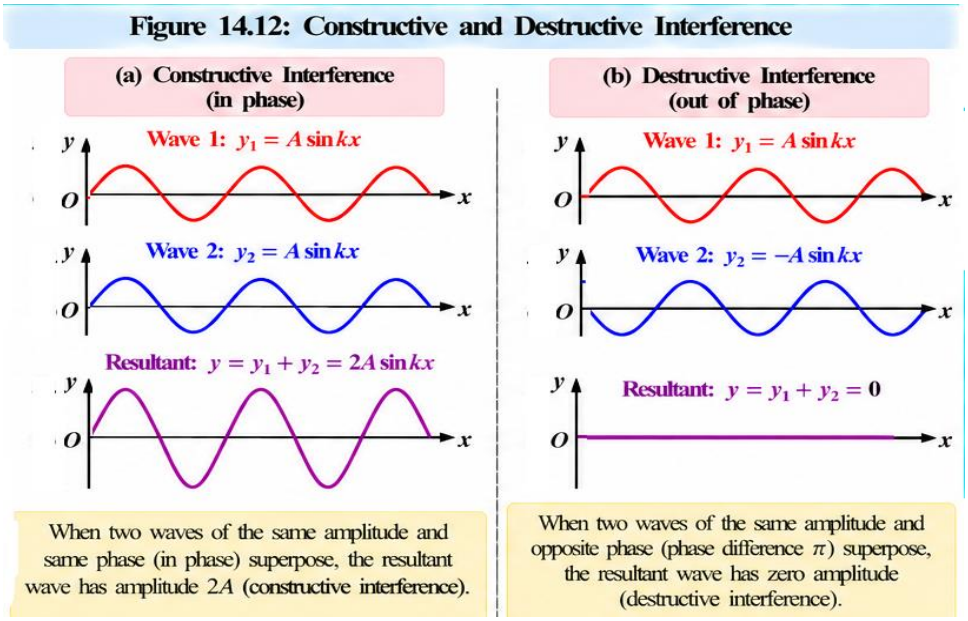
For destructive interference, cos φ = −1:

$$I_{min} = (\sqrt{I_1} - \sqrt{I_2})^2$$

Important Note

Interference redistributes energy. It does not mean that energy is permanently destroyed at points of destructive interference.

Figure 14.12: Constructive and Destructive Interference



3.4 Reflection of Waves

Definition

Reflection of waves is the phenomenon in which a wave returns into the original medium after encountering a boundary or obstacle.

Reflection may occur when a wave reaches the end of a stretched string, a rigid support, an open end, or a change in the properties of a medium.

The behaviour of the reflected wave depends on the boundary conditions.

A. Reflection at a Fixed End

When a transverse pulse travelling along a string reaches a fixed end, the reflected pulse is inverted.

A crest is reflected as a trough, and a trough is reflected as a crest.

The reflected displacement has a phase change of π radians, equivalent to 180°.

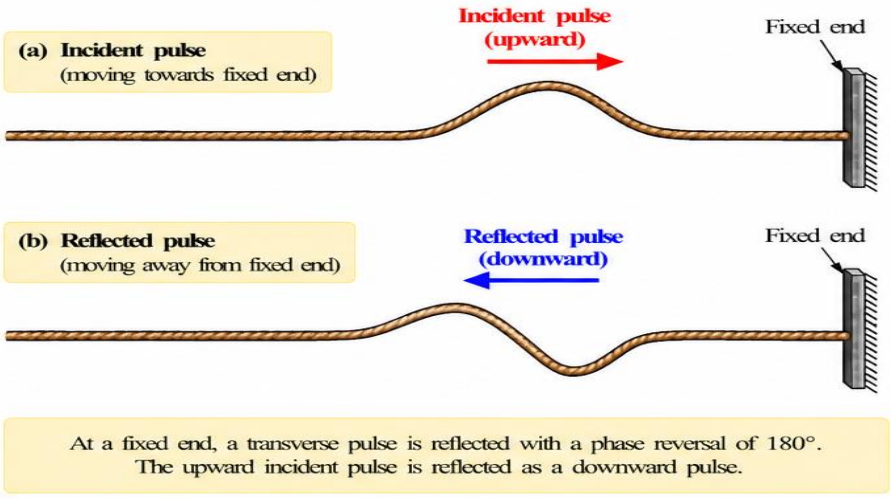
Phase change at a fixed end = π radians

This occurs because the displacement of the string at the fixed boundary must remain zero.

Example

If an upward pulse travels towards a rigidly fixed end, the reflected pulse is downward.

Figure 14.13: Reflection of a Transverse Pulse at a Fixed End



At a fixed end, a transverse pulse is reflected with a phase reversal of 180°.
The upward incident pulse is reflected as a downward pulse.

B. Reflection at a Free End

When a transverse pulse reaches an ideal free end, it is reflected without inversion.

A crest returns as a crest, and a trough returns as a trough.

The reflected wave undergoes no phase reversal at an ideal free end.

Phase change at an ideal free end = 0

The free end is able to move transversely, so its displacement is not constrained to remain zero.

Fixed End versus Free End

Property	Fixed End	Ideal Free End
Reflected pulse	Inverted	Not inverted
Phase change	π radians	0 radians
Crest reflects as	Trough	Crest
Boundary condition	Displacement is zero	Transverse force at the end is zero in the ideal model

Reflection of Sound Waves

Sound waves may also be reflected from surfaces.
Examples include:

- Echoes.
- Reverberation in rooms.
- Reflection of sound from walls and buildings.
- Sound reflection in auditoriums.

The detailed study of echoes and reverberation is an application of wave reflection.

3.5 Formation of Standing Waves

Definition

A **standing wave** also called a stationary wave, is formed when two waves of the same frequency, wavelength and amplitude travel in opposite directions through the same medium and superpose.
Standing waves are commonly produced by the interference of an incident wave and its reflected wave.
Unlike a progressive wave, a standing wave does not transfer energy continuously along the medium as a travelling wave does.

Derivation of the Standing Wave Equation

Consider two waves of equal amplitude A and angular frequency ω travelling in opposite directions.
Let the waves be: $y_1 = A \sin(kx - \omega t)$ and $y_2 = A \sin(kx + \omega t)$
The resultant displacement is: $y = y_1 + y_2$
Therefore: $y = A \sin(kx - \omega t) + A \sin(kx + \omega t)$
Using the identity: $\sin(P - Q) + \sin(P + Q) = 2 \sin P \cos Q$
we obtain: $\mathbf{y = 2A \sin kx \cos \omega t}$
This is the equation of a standing wave.
The equation can also be written in other equivalent forms depending on the choice of origin and phase.

Interpretation

$y = 2A \sin kx \cos \omega t$
The displacement depends on both position x and time t.
The amplitude at position x is: $\mathbf{A_{resultant} = 2A\,|\sin kx|}$
Thus, the amplitude varies from point to point.
At certain positions, the amplitude is always zero. At other positions, it reaches its maximum value of 2A.
These positions are called nodes and antinodes, respectively.

3.6 Nodes and Antinodes

A. Nodes

A **node** is a point in a standing wave where the resultant displacement is always zero.
From the standing wave equation: $y = 2A \sin kx \cos \omega t$
At a node: $\sin kx = 0$
Therefore: $kx = n\pi$
Since $k = 2\pi/\lambda$: Therefore: $(2\pi/\lambda)x = n\pi$
Hence: $\mathbf{x = n\lambda/2}$

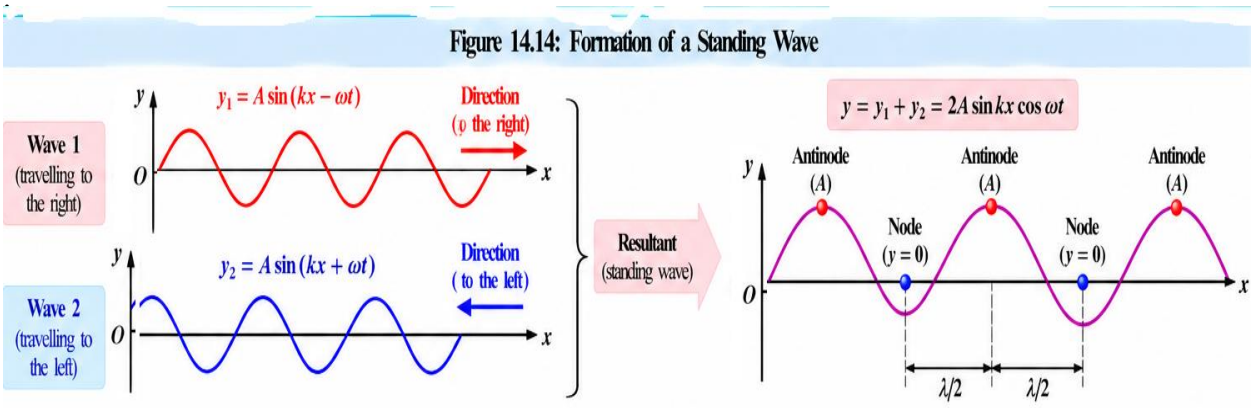
Where, n = 0, 1, 2, 3...
Thus, the positions of nodes are: x = 0, λ/2, λ, 3λ/2, 2λ, ...
Distance between two consecutive nodes = λ/2

B. Antinodes

An **antinode** is a point in a standing wave where the amplitude of oscillation is maximum.
From the standing wave equation:
Maximum amplitude occurs when: |sin kx| = 1
Therefore: $kx = (2n + 1)\pi/2$
Substituting $k = 2\pi/\lambda$: Therefore: $(2\pi/\lambda)x = (2n + 1)\pi/2$
Hence: **$x = (2n + 1)\lambda/4$**
Where, n = 0, 1, 2, 3...
The positions of antinodes are: $x = \lambda/4, 3\lambda/4, 5\lambda/4, 7\lambda/4,...$
Distance between two consecutive antinodes = λ/2
The distance between a node and its nearest antinode is:
Distance between a node and its nearest antinode = λ/4

Important Properties of Standing Waves

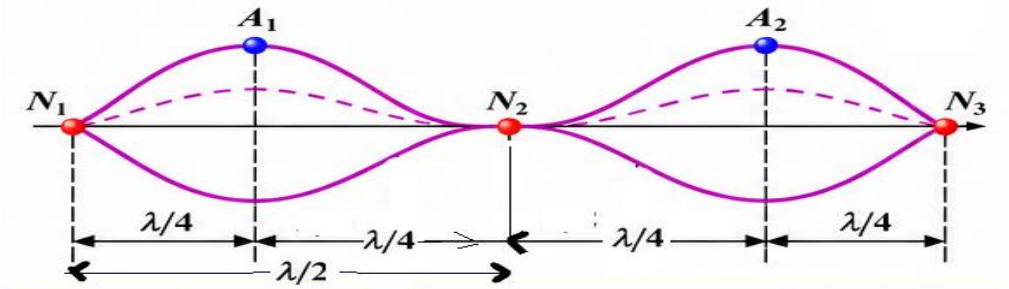
1. Nodes remain permanently at rest in an ideal standing wave.
2. Antinodes oscillate with maximum amplitude.
3. All particles between two adjacent nodes oscillate with the same frequency.
4. Particles in adjacent segments separated by a node oscillate in opposite phases.
5. The amplitude varies with position.
6. There is no net continuous transfer of energy along an ideal standing wave.
7. The distance between adjacent nodes is λ/2.
8. The distance between adjacent antinodes is λ/2.
9. The distance between a node and its nearest antinode is λ/4.



3.7 Progressive Waves versus Standing Waves

Property	Progressive Wave	Standing Wave
Formation	Travels through the medium	Formed by superposition of opposite-travelling waves
Energy transfer	Transfers energy along the medium	No net continuous energy transfer in the ideal case
Amplitude	Constant for an ideal undamped wave	Varies with position
Nodes	No permanent nodes in a uniform travelling sinusoidal wave	Permanent nodes are present
Antinodes	No permanent antinodes	Permanent antinodes are present
Phase relationship	Phase changes continuously with position	Particles within one segment between adjacent nodes oscillate in the same phase
Adjacent segments	Not applicable in this form	Oscillate in opposite phases
Example	Wave travelling along a string	Vibrating string fixed at both ends

Figure 14.15: Nodes and Antinodes in a Standing Wave



In a standing wave, the points of zero displacement are called nodes (N_1, N_2, N_3) and the points of maximum displacement are called antinodes (A_1, A_2). The distance between two adjacent nodes is $\lambda/2$ and between a node and the nearest antinode is $\lambda/4$.

3.8 Solved Numerical Problems

Example 1: Resultant Amplitude

Two waves of amplitudes 3 cm and 4 cm travel through the same medium with a phase difference of 60°. Calculate the resultant amplitude.

Solution

Given: $A_1 = 3$ cm, $A_2 = 4$ cm, $\varphi = 60°$

The resultant amplitude is:

$$A = \sqrt{(A_1^2 + A_2^2 + 2A_1A_2 \cos \varphi)}$$

Substituting:

$$A = \sqrt{[3^2 + 4^2 + 2 \times 3 \times 4 \times \cos 60°]}$$

Since $\cos 60° = 1/2$:

$$A = \sqrt{(9 + 16 + 12)} = \sqrt{37}$$

Answer: $A \approx 6.08$ cm

Example 2: Constructive Interference

Two coherent waves of equal amplitude 2 cm interfere constructively. Find the resultant amplitude.

Solution

Given: $A_1 = A_2 = 2$ cm,

For constructive interference: $A = A_1 + A_2 = 2 + 2$

Answer: A = 4 cm

Example 3: Destructive Interference

Two coherent waves of amplitudes 5 cm and 3 cm interfere with a phase difference of π radians. Find the resultant amplitude.

Solution

For destructive interference: $A = |A_1 - A_2| = |5 - 3|$ **= 2 cm**

The resultant amplitude is not zero because the individual amplitudes are unequal.

Example 4: Intensity in Constructive Interference

Two coherent waves each have an individual intensity I_0. Find the maximum resultant intensity.

Solution

For equal individual intensities: $I_{max} = (\sqrt{I_0} + \sqrt{I_0})^2 = (2\sqrt{I_0})^2$ **$= 4I_0$**

Example 5: Positions of Nodes

A standing wave has a wavelength of 0.80 m. Find the distance between two consecutive nodes.

Solution

The distance between consecutive nodes is:

$$d = \lambda/2 = 0.80/2 = \mathbf{0.40\ m}$$

Example 6: Distance Between a Node and an Antinode

The wavelength of a standing wave is 1.20 m. Calculate the distance between a node and its nearest antinode.

Solution

Distance $= \lambda/4 = 1.20/4$ **= 0.30 m**

Example 7: Phase Difference and Path Difference

Two coherent waves have a wavelength of 0.50 m and a path difference of 0.125 m. Calculate their phase difference.

Solution

The phase difference is:

$$\varphi = 2\pi\Delta x/\lambda$$

Substituting: $\varphi = 2\pi \times 0.125/0.50 = \pi/2$ radians

Answer: $\varphi = \pi/2$ radians = 90°

3.9 Important Conceptual Questions

Q1. What is the principle of superposition?

The resultant displacement due to overlapping waves is the algebraic sum of the individual displacements.

Q2. What is the condition for constructive interference?

The path difference must be an integral multiple of the wavelength:

$$\Delta x = n\lambda$$

Q3. What is the condition for destructive interference?
The path difference must be an odd multiple of half the wavelength:

$\Delta x = (2n + 1)\lambda/2$

Q4. Why is the amplitude of a standing wave not the same everywhere?
Because the resultant amplitude depends on position according to A_resultant = 2A|sin kx| for the chosen standing-wave equation.

Q5. What is a node?
A node is a point where the resultant displacement remains zero at all times.

Q6. What is an antinode?
An antinode is a point where the oscillation amplitude is maximum.

Q7. What is the distance between adjacent nodes?

$\lambda/2$.

Q8. What is the distance between a node and its nearest antinode?

$\lambda/4$.

Q9. What happens when a transverse pulse reflects from a fixed end?
It is reflected with inversion, corresponding to a phase change of π radians.

Q10. Does a standing wave transfer energy continuously along the medium?
An ideal standing wave has no net continuous energy transfer along the medium.

PART 4: STANDING WAVES IN STRINGS AND ORGAN PIPES, FUNDAMENTAL FREQUENCY, HARMONICS, OVERTONES AND BEATS

4.1 Standing Waves in a Stretched String

When a stretched string is fixed at both ends and set into vibration, transverse waves travel along the string and are reflected at the fixed ends.
The incident and reflected waves superpose to form standing waves.
Examples include:

- Vibrating strings of a guitar.
- Strings of a violin.
- Strings of a sitar.
- Vibrating strings of a piano.

Since both ends of the string are fixed, the displacement at each end must always be zero.
Therefore, **both ends of the string are nodes**.
The allowed standing-wave patterns depend on the length of the string and its boundary conditions.

Wave Speed on a Stretched String

The speed of a transverse wave on a stretched string is: $\mathbf{v = \sqrt{(T/\mu)}}$
Where:

- v = Speed of the transverse wave.
- T = Tension in the string.
- μ = Linear mass density of the string.

Linear mass density is: $\mathbf{\mu = m/L}$
Where m is the mass and L is the length of the string.
Hence,

$$\mathbf{v = \sqrt{(TL/m)}}$$

This speed is determined by the tension and linear mass density of the string.

4.2 Fundamental Frequency of a Stretched String

The **fundamental mode** is the simplest possible standing-wave pattern in a string fixed at both ends.
In the fundamental mode:

- Both ends are nodes.
- One antinode lies at the centre.
- The string contains half a wavelength.

Therefore, $L = \lambda_1/2$
Hence, $\mathbf{\lambda_1 = 2L}$
The frequency of a wave is: $f = v/\lambda$
Therefore, the fundamental frequency is: $f_1 = v/(2L)$
Substituting $v = \sqrt{(T/\mu)}$:

$$\mathbf{f_1 = (1/2L)\sqrt{(T/\mu)}}$$

This is the fundamental-frequency formula for a string fixed at both ends.

Important Conclusions

From the formula: $f_1 = (1/2L)\sqrt{(T/\mu)}$

We conclude that:

1. Fundamental frequency is inversely proportional to the length of the string.
2. Fundamental frequency is directly proportional to the square root of tension.
3. Fundamental frequency is inversely proportional to the square root of linear mass density.

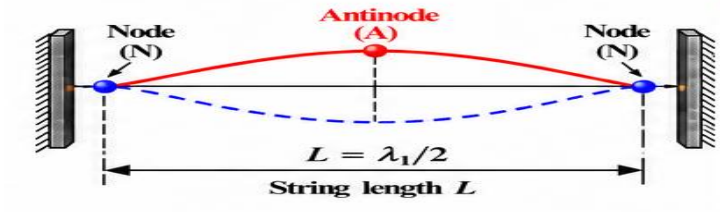
Thus,

$$f_1 \propto 1/L$$
$$f_1 \propto \sqrt{T}$$
$$f_1 \propto 1/\sqrt{\mu}$$

Physical Interpretation

- A shorter string produces a higher frequency.
- Increasing the tension increases the frequency.
- A string with lower linear mass density produces a higher frequency when its length and tension remain unchanged.

Figure 14.16: Fundamental Mode of a Stretched String



Fundamental mode (first harmonic) of a stretched string fixed at both ends. The ends are nodes and the centre is an antinode. The length of the string is $L = \lambda_1/2$.

4.3 Harmonics of a Stretched String

For a string fixed at both ends, the allowed standing-wave patterns satisfy the condition that the length contains an integral number of half-wavelengths.

For the nth harmonic: $\mathbf{L = n\lambda_n/2}$

Where n = 1, 2, 3, ...

Therefore,

$$\lambda_n = 2L/n$$

The frequency of the nth harmonic is:

$$f_n = v/\lambda_n$$

Substituting $\lambda_n = 2L/n$:

$$f_n = nv/(2L)$$

Since $f_1 = v/(2L)$:

$$f_n = nf_1$$

Thus, the frequencies of the harmonics are:

- First harmonic: f_1
- Second harmonic: $f_2 = 2f_1$
- Third harmonic: $f_3 = 3f_1$
- Fourth harmonic: $f_4 = 4f_1$
- nth harmonic: $f_n = nf_1$

Wavelengths of the First Three Harmonics

First harmonic: $\lambda_1 = 2L$
Second harmonic: $\lambda_2 = L$
Third harmonic: $\lambda_3 = 2L/3$

Nodes and Antinodes

For the nth harmonic of a string fixed at both ends:

- There are n antinodes.
- There are n + 1 nodes, including both endpoints.

Figure 14.16-1: First Three Harmonics of a Stretched String

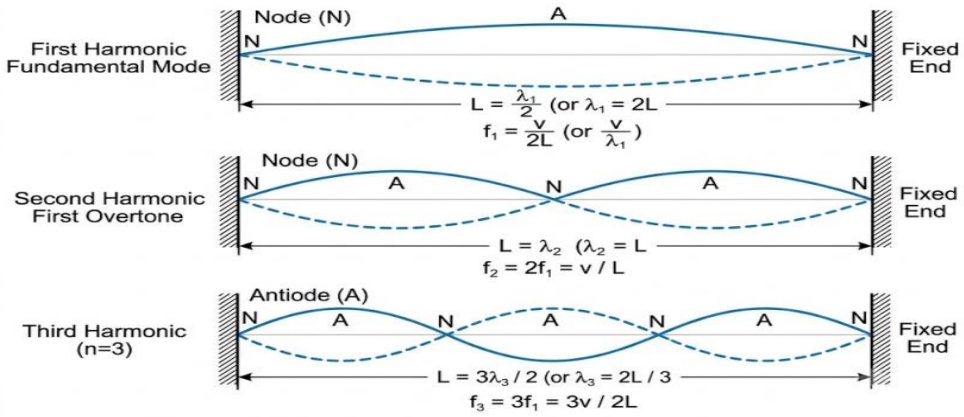


Figure 14.16-1: First Three Harmonics of a Stretched String

For example:

Harmonic	Frequency	Wavelength	Number of Antinodes	Number of Nodes
First	f_1	2L	1	2
Second	$2f_1$	L	2	3
Third	$3f_1$	2L/3	3	4
Fourth	$4f_1$	L/2	4	5
nth	nf_1	2L/n	n	n + 1

4.4 Harmonics and Overtones

Harmonics

A harmonic is a frequency that is an integral multiple of the fundamental frequency.
For a string fixed at both ends: $\mathbf{f_n = nf_1}$
Therefore, all integer harmonics are present among the allowed ideal modes of vibration.

Overtones

An overtone is a frequency above the fundamental frequency.
The first overtone is the lowest-frequency mode above the fundamental.
For a string fixed at both ends:

- Fundamental frequency = f_1
- First overtone = $f_2 = 2f_1$
- Second overtone = $f_3 = 3f_1$
- Third overtone = $f_4 = 4f_1$

For this string, the first overtone is the second harmonic, and the second overtone is the third harmonic.
Important: Harmonic number and overtone number are not the same. The fundamental is the first harmonic but is not an overtone.

4.5 Standing Waves in Open Organ Pipes

An **open organ pipe** is a pipe whose two ends are open to the atmosphere.
Examples include some wind instruments and open-ended laboratory resonance tubes.
For sound waves in an open pipe:

- The air at an open end can move with relatively large displacement.
- An ideal open end is treated as a displacement antinode.
- Pressure variations are minimal at the open end, making it a pressure node.

Thus, for the ideal open-pipe model, **both ends are displacement antinodes**.
The allowed standing-wave patterns must satisfy the boundary conditions at both ends.

Fundamental Frequency of an Open Pipe

In the fundamental mode, the pipe contains half a wavelength.
Therefore, $L = \lambda_1/2$
Hence, $\mathbf{\lambda_1 = 2L}$
The fundamental frequency is: $f_1 = v/\lambda_1$
Therefore, $\mathbf{f_1 = v/(2L)}$
Where v is the speed of sound in air and L is the length of the pipe.

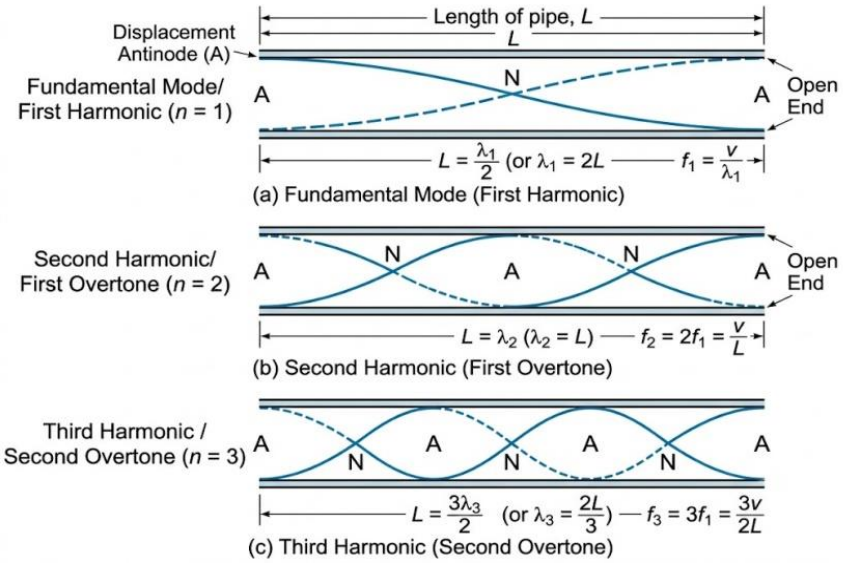
Harmonics of an Open Pipe

For the nth harmonic: $L = n\lambda_n/2$
Therefore, $\mathbf{\lambda_n = 2L/n}$
The frequency is: $\mathbf{f_n = nv/(2L)}$
Hence, $\mathbf{f_n = nf_1}$
The frequencies of an ideal open pipe are:
$f_1 = v/(2L)$
$f_2 = v/L = 2f_1$
$f_3 = 3v/(2L) = 3f_1$
$f_4 = 2v/L = 4f_1$
Thus, an ideal open pipe supports all integer harmonics.

Important Properties of an Open Pipe

1. Both ends are displacement antinodes.
2. The fundamental mode contains half a wavelength.
3. The fundamental frequency is $f_1 = v/(2L)$.
4. All integer harmonics are allowed in the ideal model.
5. For a given speed of sound, increasing the pipe length decreases its fundamental frequency.

Figure 14.17: Fundamental Mode and Harmonics of an Open Organ Pipe



4.6 Standing Waves in Closed Organ Pipes

A **closed organ pipe** is a pipe that is closed at one end and open at the other.
For an ideal closed pipe:

- The closed end is a displacement node.
- The open end is a displacement antinode.

The pressure pattern is opposite:

- The closed end is a pressure antinode.
- The open end is a pressure node.

Fundamental Frequency of a Closed Pipe

In the fundamental mode, the pipe contains one-quarter of a wavelength.
Therefore, $L = \lambda_1/4$
Hence, $\boldsymbol{\lambda_1 = 4L}$
The fundamental frequency is: $f_1 = v/\lambda_1$
Therefore, $\boldsymbol{f_1 = v/(4L)}$
This is the fundamental frequency formula for an ideal closed pipe.

Harmonics of a Closed Pipe

For a closed pipe, the displacement must be zero at the closed end and maximum at the open end.
The allowed patterns satisfy:

$$L = (2n - 1)\lambda_n/4$$

Where n = 1, 2, 3...
Therefore, $\boldsymbol{\lambda_n = 4L/(2n - 1)}$
The corresponding frequencies are:

$$\boldsymbol{f_n = (2n - 1)v/(4L)}$$

Since $f_1 = v/(4L)$:

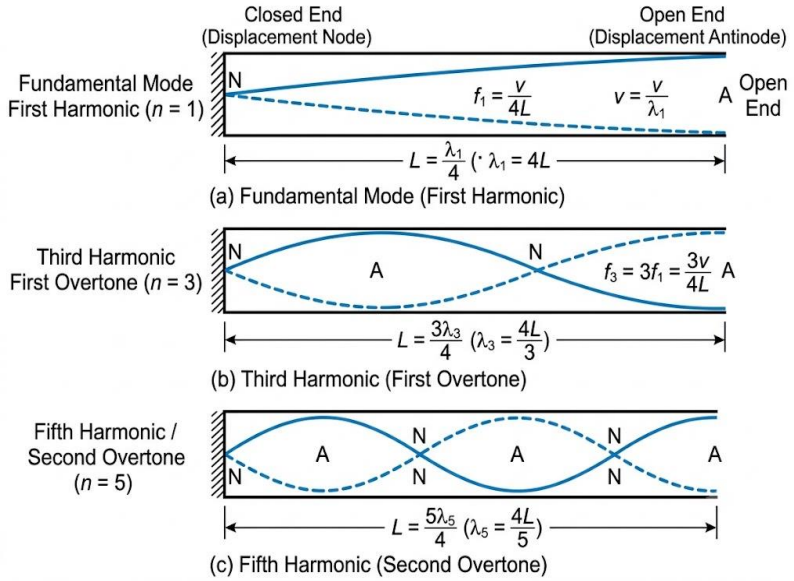
$$\boldsymbol{f_n = (2n - 1)f_1}$$

Thus, the allowed frequencies are:

- Fundamental: f_1
- Third harmonic: $3f_1$

- Fifth harmonic: $5f_1$
- Seventh harmonic: $7f_1$

Figure 14.18: Fundamental Mode and Odd Harmonics of a Closed Organ Pipe



Only odd harmonics are present in the ideal closed pipe.

Why Are Even Harmonics Absent?

The boundary conditions require a displacement node at the closed end and a displacement antinode at the open end.

A pattern with an even harmonic number cannot satisfy both of these conditions simultaneously for a pipe of fixed length.

Therefore, the ideal closed pipe supports only odd harmonics.

Important Properties of a Closed Pipe

1. The closed end is a displacement node.
2. The open end is a displacement antinode.
3. The fundamental wavelength is 4L.
4. The fundamental frequency is $f_1 = v/(4L)$.
5. Only odd harmonics occur in the ideal model.
6. The first overtone is the third harmonic, with frequency $3f_1$.
7. The second overtone is the fifth harmonic, with frequency $5f_1$.

4.7 Comparison of Open and Closed Organ Pipes

Property	Open Pipe	Closed Pipe
Boundary condition	Both ends open	One end closed, one end open
Displacement at the open end	Antinode	Antinode
Displacement at the closed end	Not applicable	Node
Fundamental wavelength	$\lambda_1 = 2L$	$\lambda_1 = 4L$
Fundamental frequency	$f_1 = v/(2L)$	$f_1 = v/(4L)$
Harmonics	All integer harmonics	Only odd harmonics
First overtone	$2f_1$	$3f_1$
Second overtone	$3f_1$	$5f_1$
Fundamental frequency for equal lengths	Open pipe-Higher	Closed pipe-Lower

For ideal pipes of equal length, with the same speed of sound:

$$f(\text{open}) = v/(2L)$$

$$f(\text{closed}) = v/(4L)$$

Therefore,

$$\mathbf{f(open) = 2f(closed)}$$

The fundamental frequency of an ideal open pipe is twice that of an ideal closed pipe of the same length.

4.8 End Correction in Organ Pipes

In a real organ pipe, the displacement antinode at an open end lies slightly outside the physical end of the pipe.
Consequently, the effective acoustic length is slightly greater than the actual length.
This additional length is called the **end correction**.
For a pipe with one open end, a common approximate result is: $\mathbf{e \approx 0.6r}$
Where:

- e = End correction at one open end.
- r = Internal radius of the pipe.

This approximation applies to a circular pipe opening under suitable conditions.

Effective Length of an Open Pipe

For a pipe open at both ends, if each end has approximately the same correction e:

$$\mathbf{L(eff) = L + 2e}$$

Its approximate fundamental frequency is:

$$\mathbf{f_1 \approx v/[2(L + 2e)]}$$

Effective Length of a Closed Pipe

For a pipe closed at one end and open at the other:

$$\mathbf{L(eff) = L + e}$$

Its approximate fundamental frequency is:

$$\mathbf{f_1 \approx v/[4(L + e)]}$$

End correction is an important practical refinement. In elementary problems, it should be included only when the question specifies it or asks for a more accurate result.

4.9 Beats

Definition

Beats are periodic variations in the intensity or loudness of sound produced when two sound waves of slightly different frequencies superpose.

Beats are heard when two sound sources produce frequencies close enough for the changes in loudness to be distinguished.

Examples include:

- Tuning a musical instrument.
- Comparing the frequencies of two tuning forks.
- Adjusting the pitch of two musical notes.

Derivation of the Beat Frequency

Consider two sound waves of equal amplitude and slightly different frequencies:

$$y_1 = A \sin(2\pi f_1 t)$$

$$y_2 = A \sin(2\pi f_2 t)$$

The resultant displacement is: $y = y_1 + y_2$

Therefore: $y = A \sin(2\pi f_1 t) + A \sin(2\pi f_2 t)$

Using the trigonometric identity: $\sin P + \sin Q = 2 \sin[(P + Q)/2] \cos[(P - Q)/2]$

We obtain: $y = 2A \sin[\pi(f_1 + f_2)t] \cos[\pi(f_1 - f_2)t]$

The amplitude envelope varies according to: $\mathbf{A_{resultant} = 2A\,|\cos[\pi(f_1 - f_2)t]|}$

The loudness rises and falls periodically.

The number of beats per second is: $\mathbf{f(beat) = |f_1 - f_2|}$

This is the beat-frequency formula.

Important Points

1. Beats arise because the relative phase of the two waves changes with time.
2. When the waves reinforce each other, the sound becomes louder.
3. When they partially or completely cancel, the sound becomes quieter.
4. The beat frequency equals the absolute difference between the two frequencies.
5. For two equal-amplitude waves, the ideal resultant amplitude periodically falls to zero.
6. For unequal amplitudes, the minimum amplitude is generally non-zero.

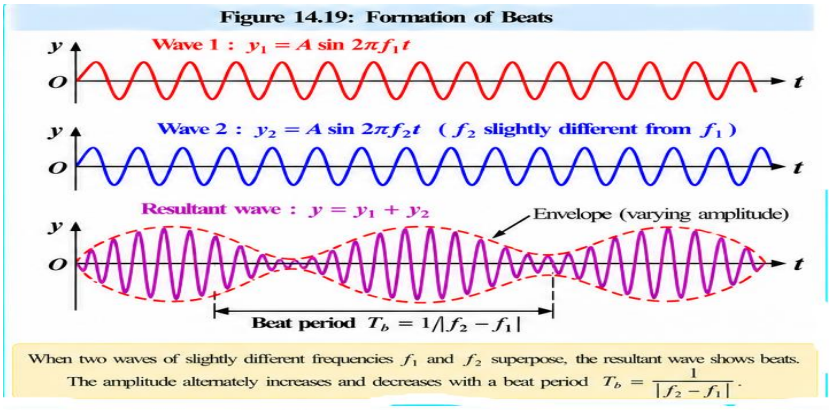
Example

Two tuning forks have frequencies of 256 Hz and 260 Hz.

The beat frequency is:

$$f(beat) = |260 - 256| = \mathbf{4\ Hz}$$

Therefore, four beats are heard per second.



4.10 Beat Period

The **beat period** is the time interval between two consecutive maxima of loudness.
It is the reciprocal of the beat frequency.

Therefore: **$T(beat) = 1/f(beat)$**
Since: $f(beat) = |f_1 - f_2|$
We obtain: **$T(beat) = 1/|f_1 - f_2|$**

Example

Two sound sources produce 5 beats per second.
The beat frequency is: $f(beat) = 5$ Hz
Therefore,

$$T(beat) = 1/5$$

Answer: $T(beat) = 0.20$ s
The loudness reaches successive maxima at intervals of 0.20 s.

4.11 Applications of Beats

1. Tuning Musical Instruments

A musician compares the sound of an instrument with a reference note.

- If beats are heard, the frequencies are different.
- If the beat frequency decreases, the frequencies are approaching one another.
- When the frequencies become equal, the beats disappear, assuming stable sources and suitable listening conditions.

2. Finding an Unknown Frequency

If one frequency is known and the beat frequency is measured, the unknown frequency may be determined.
If the known frequency is f_1 and the beat frequency is $f(beat)$:

$$\mathbf{f_2 = f_1 \pm f(beat)}$$

Both possibilities must initially be considered. Additional information is needed to determine whether the unknown frequency is above or below the known frequency.

3. Comparing Two Tuning Forks

Two tuning forks with slightly different frequencies produce beats. Counting the beats per second gives the difference between their frequencies.

4.12 Solved Numerical Problems

Example 1: Fundamental Frequency of a Stretched String

A string of length 1.0 m is stretched under a tension of 100 N. Its linear mass density is 0.01 kg/m. Calculate its fundamental frequency.

Solution

Given: $L = 1.0$ m, $T = 100$ N, $\mu = 0.01$ kg/m
First, calculating the wave speed:

$$v = \sqrt{(T/\mu)}\ v = \sqrt{(100/0.01)} = \sqrt{10000} = 100 \text{ m/s}$$

The fundamental frequency is:

$$f_1 = v/(2L) = 100/(2 \times 1.0)$$

Answer: $f_1 = 50$ Hz

Example 2: Frequency of the Third Harmonic
A string has a fundamental frequency of 120 Hz. Calculate its third harmonic frequency.
Solution
For a string fixed at both ends:

$$f_n = nf_1$$

For the third harmonic, n = 3.

$$f_3 = 3 \times 120$$

Answer: $f_3 = 360$ Hz
Example 3: Fundamental Frequency of an Open Pipe
An open organ pipe has a length of 0.85 m. The speed of sound is 340 m/s. Calculate its fundamental frequency, neglecting end correction.
Solution
Given: L = 0.85 m, v = 340 m/s,
For an open pipe:

$$f_1 = v/(2L) = 340/(2 \times 0.85) = 340/1.70$$

Answer: $f_1 = 200$ Hz

Example 4: Fundamental Frequency of a Closed Pipe
A closed organ pipe has a length of 0.85 m. The speed of sound is 340 m/s. Calculate its fundamental frequency, neglecting end correction.
Solution
For a closed pipe:

$$f_1 = v/(4L) = 340/(4 \times 0.85) = 340/3.40$$

Answer: $f_1 = 100$ Hz
Notice that the open pipe of the same length has twice the fundamental frequency of the closed pipe.

Example 5: First Overtone of a Closed Pipe
The fundamental frequency of a closed pipe is 150 Hz. Calculate its first overtone.
Solution
A closed pipe supports odd harmonics only.
Its first overtone is the third harmonic.

$$f \text{ (first overtone)} = 3f_1 = 3 \times 150$$

Answer: 450 Hz

Example 6: Beat Frequency
Two tuning forks have frequencies of 512 Hz and 518 Hz. Calculate the beat frequency.
Solution

$$f(\text{beat}) = |f_1 - f_2| = |518 - 512|$$

Answer: f(beat) = 6 Hz
Six beats are heard per second.

Example 7: Unknown Frequency Using Beats
A tuning fork of frequency 256 Hz produces 4 beats per second with another tuning fork. Find the possible frequency of the second tuning fork.

Solution

$$f(\text{beat}) = |f_1 - f_2|$$

Or,

$$4 = |256 - f_2|$$

Therefore, the possible frequencies are:

$$f_2 = 256 + 4 = 260 \text{ Hz}$$

Or, $f_2 = 256 - 4 = 252$ Hz

Answer: $f_2 = 252$ Hz or 260 Hz

Without additional information, both answers are possible.

Example 8: Beat Period

Two sound waves produce 8 beats per second. Calculate the beat period.

Solution

$$f(\text{beat}) = 8 \text{ Hz}$$

$$T(\text{beat}) = 1/f(\text{beat}) = 1/8$$

Answer: T(beat) = 0.125 s

4.13 Important Conceptual Questions

Q1. Why are both ends of a stretched string nodes?

Because the string is fixed at both ends, so the transverse displacement at each endpoint must remain zero.

Q2. What is the fundamental frequency of a string fixed at both ends?

$f_1 = v/(2L) = (1/2L)\sqrt{(T/\mu)}$.

Q3. Which harmonics are present in an ideal string fixed at both ends?

All integer harmonics: $f_n = nf_1$.

Q4. What is the fundamental wavelength of an open organ pipe?

$\lambda_1 = 2L$.

Q5. What is the fundamental wavelength of a closed organ pipe?

$\lambda_1 = 4L$.

Q6. Why does an ideal closed pipe support only odd harmonics?

Its boundary conditions require a displacement node at the closed end and a displacement antinode at the open end. Only odd harmonics satisfy these conditions.

Q7. What is the first overtone of a closed pipe?

The third harmonic with frequency $3f_1$ is the first overtone of a closed pipe.

Q8. What are beats?

Beats are periodic variations in the loudness of sound produced by the superposition of two waves of slightly different frequencies.

Q9. What is the beat-frequency formula?

$f \text{ beat} = |f_1 - f_2|$.

Q10. Can beat frequency determine an unknown frequency uniquely?

Not always. If one frequency is known, the unknown frequency may be either f known + f beat or f known – f beat.
